

On the Reynolds number expansion for the Navier-Stokes equations

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Abstract

In a previous paper of ours [14] we have considered the incompressible Navier-Stokes (NS) equations on a d -dimensional torus \mathbf{T}^d , in the functional setting of the Sobolev spaces $\mathbb{H}_{\Sigma_0}^n(\mathbf{T}^d)$ of divergence free, zero mean vector fields ($n > d/2 + 1$). In the cited work we have presented a general setting for the a posteriori analysis of approximate solutions of the NS Cauchy problem; given any approximate solution u_a , this allows to infer a lower bound T_c on the time of existence of the exact solution u and to construct a function \mathcal{R}_n such that $\|u(t) - u_a(t)\|_n \leq \mathcal{R}_n(t)$ for all $t \in [0, T_c)$. In certain cases it is $T_c = +\infty$, so global existence is granted for u . In the present paper the framework of [14] is applied using as an approximate solution an expansion $u^N(t) = \sum_{j=0}^N R^j u_j(t)$, where R is the Reynolds number. This allows, amongst else, to derive the global existence of u when R is below some critical value R_* (increasing with N in the examples that we analyze). After a general discussion about the Reynolds expansion and its a posteriori analysis, we consider the expansions of orders $N = 1, 2, 5$ in dimension $d = 3$, with the initial datum of Behr, Nečas and Wu [1]. Computations of order $N = 5$ yield a quantitative improvement of the results previously obtained for this initial datum in [14], where a Galerkin approximate solution was employed in place of the Reynolds expansion.

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1 Introduction

The incompressible Navier-Stokes (NS) equations with no external forces and periodic boundary conditions can be written as

$$\frac{\partial u}{\partial \mathfrak{t}} = \nu \Delta u + \mathcal{P}(u, u) , \quad (1.1)$$

where: $\nu \in (0, +\infty)$ is the viscosity coefficient; $u = u(x, \mathfrak{t})$ is the divergence free velocity field; the space variables $x = (x_s)_{s=1, \dots, d}$ belong to the torus \mathbf{T}^d (and yield the derivatives $\partial_s := \partial/\partial x_s$); $\Delta := \sum_{s=1}^d \partial_{ss}$ is the Laplacian. Furthermore, \mathcal{P} is the bilinear map defined as follows: for all sufficiently regular velocity fields v, w on \mathbf{T}^d ,

$$\mathcal{P}(v, w) := -\mathfrak{L}(v \bullet \partial w) \quad (1.2)$$

where $(v \bullet \partial w)_r := \sum_{s=1}^d v_s \partial_s w_r$ ($r = 1, \dots, d$), and \mathfrak{L} is the Leray projection onto the space of divergence free vector fields. The dimension d is arbitrary in the general setting of this paper, but we put $d = 3$ in the application of the last section. From the physical viewpoint, the velocity fields (at any fixed time) should be maps from \mathbf{T}^d to \mathbf{R}^d ; however, we can harmlessly consider maps $\mathbf{T}^d \rightarrow \mathbf{C}^d$.

Let us introduce the rescaled time t and the Reynolds number R , setting

$$t := \nu \mathfrak{t} , \quad R := \frac{1}{\nu} ; \quad (1.3)$$

then Eq (1.1) takes the form

$$\frac{\partial u}{\partial t} = \Delta u + R \mathcal{P}(u, u) , \quad (1.4)$$

that will be the standard of this paper. Our functional setting for Eq. (1.4) is based on the Sobolev spaces

$$\mathbb{H}_{\Sigma_0}^n(\mathbf{T}^d) \equiv \mathbb{H}_{\Sigma_0}^n := \{v : \mathbf{T}^d \rightarrow \mathbf{C}^d \mid \langle v \rangle = 0, \operatorname{div} v = 0, \sqrt{-\Delta}^n v \in \mathbb{L}^2(\mathbf{T}^d)\} \quad (1.5)$$

(where $\langle \cdot \rangle$ indicates the mean over \mathbf{T}^d); for any real n , the above space is equipped with the inner product $\langle v | w \rangle_n := \langle \sqrt{-\Delta}^n v | \sqrt{-\Delta}^n w \rangle_{L^2}$ and with the corresponding norm $\| \cdot \|_n$.

In our paper [14] we have outlined a general framework to obtain quantitative information on the exact solution u of the NS Cauchy problem analyzing *a posteriori* an approximate solution. To be more precise, consider the NS equation (1.4) with a specified initial condition $u(x, 0) = u_*(x)$; let $u_a : \mathbf{T}^d \times [0, T_a) \rightarrow \mathbf{C}^d$ be an approximate solution of this Cauchy problem, and consider (for $n > d/2 + 1$) the Sobolev norms

$$\left\| \left(\frac{\partial u_{\mathbf{a}}}{\partial t} - \Delta u_{\mathbf{a}} - R \mathcal{P}(u_{\mathbf{a}}, u_{\mathbf{a}}) \right)(t) \right\|_n, \quad \|u_{\mathbf{a}}(0) - u_*\|_n, \quad (1.6)$$

$$\|u_{\mathbf{a}}(t)\|_n, \quad \|u_{\mathbf{a}}(t)\|_{n+1}, \quad (1.7)$$

where $t \in [0, T_{\mathbf{a}})$ and $u_{\mathbf{a}}(t) := u_{\mathbf{a}}(\cdot, t)$. The norms in (1.6) control the *differential and datum errors* of $u_{\mathbf{a}}$, while the norms in (1.7) refer to the growth of $u_{\mathbf{a}}$. The approach of [14] relies on the so-called *control inequalities*; these consist of a differential inequality and of an inequality on the initial value, determined by the norms (1.6) (1.7) and involving an unknown function $\mathcal{R}_n : [0, T_c) \rightarrow [0, +\infty)$. Assume the control inequalities to have a solution \mathcal{R}_n , with a suitable domain $[0, T_c)$; then, according to [14], the solution u of the NS equation (1.4) with initial datum u_* exists (in a classical sense) on the time interval $[0, T_c)$, and its distance from the approximate solution admits the bound

$$\|u(t) - u_{\mathbf{a}}(t)\|_n \leq \mathcal{R}_n(t) \quad \text{for } t \in [0, T_c). \quad (1.8)$$

(For similar statements on the NS equations or other nonlinear evolutionary PDEs, see [3] [10] [11] [12] [18].)

In the present paper we apply the above framework choosing as an approximate solution a polynomial in R of the form

$$u^N(t) := \sum_{j=0}^N R^j u_j(t), \quad (1.9)$$

where the terms $u_j(t)$ are determined requiring the differential error to be $O(R^{N+1})$ for $R \rightarrow 0$. We emphasize that, in our approach, the order N could be large but is fixed; so we are not considering the $N \rightarrow +\infty$ limit, i.e., the solution of the Cauchy problem for Eq. (1.4) via a power series $u(t) = \sum_{j=0}^{+\infty} R^j u_j(t)$. A theoretical analysis of the convergence issue for such a series, in suitable function spaces, has been developed by some authors, especially Cannone [2] and Sinai [19]. However, the approaches of these authors yield convergence conditions (local or global in time) which depend on the initial datum u_* only through its norm; moreover, these authors have dedicated little attention to the strictly quantitative aspects of their analysis (such as the evaluations of the constants in certain inequalities). On the contrary, our approach based on a finite order approximant u^N as in (1.9) has the following features.

- (i) We produce estimates on the interval of existence of the exact solution u (and on its distance from u^N) which depend on the fine structure of the initial datum u_* and not only on its norm; the specific features of the initial datum yielding these estimates are encoded in the expression of the differential error $\partial u^N / \partial t - \Delta u^N - R \mathcal{P}(u^N, u^N)$.

- (ii) Our analysis is fully quantitative: it relies on explicit expressions for u^N and its errors and uses, amongst else, the estimates of [13] [15] on the constants in certain inequalities about \mathcal{P} .

The above setting invites a computer assisted approach: this can be readily set up when the NS initial datum u_* is sufficiently simple, say, a Fourier polynomial. In this case the approximant u^N and the errors $\partial u^N / \partial t - \Delta u^N - R\mathcal{P}(u^N, u^N)$, $u^N(0) - u_*$, with their Sobolev norms, can be determined via any package for symbolic computation; after this a solution \mathcal{R}_n for the control inequalities can be obtained numerically. More precisely, one can try to satisfy them as equalities: this amounts to solve the Cauchy problem for a simple ODE in the unknown function $t \mapsto \mathcal{R}_n(t)$. This “control Cauchy problem” is easily treated numerically.

In the present paper the above procedure is described in general terms and then applied with $d = 3$ and $n = 3$, choosing for u_* the so-called Behr-Nečas-Wu (BNW) initial datum [1]. For the practical implementation we use MATHEMATICA on a PC; first we work with $N = 1$ to introduce the method, and then pass to the orders $N = 2$, $N = 5$.

In all the above cases, the control Cauchy problem has a solution \mathcal{R}_3 of domain $[0, +\infty)$ if R is below some critical value R_* (depending on N); in this situation we can grant global existence for the solution u of the NS Cauchy problem, and the inequality $\|u(t) - u^N(t)\|_3 \leq \mathcal{R}_3(t)$ holds for all $t \in [0, +\infty)$. For R above R_* , the control problem has a solution \mathcal{R}_3 on a bounded interval $[0, T_c)$; so, the existence of the solution u of the NS Cauchy problem and the inequality $\|u(t) - u^N(t)\|_3 \leq \mathcal{R}_3(t)$ are granted only on this interval. Passing from $N = 1$ to $N = 2$, and from $N = 2$ to $N = 5$, the critical value R_* increases.

In the final part of the paper the outcomes of the above Reynolds expansions are compared with the results of [14], where an approximate NS solution was constructed for the BNW initial datum using the Galerkin method with a set of 150 Fourier modes; it turns out that this Galerkin approach is quantitatively equivalent to the Reynolds expansion of order $N = 2$, while the expansion of order $N = 5$ gives much better results concerning the global existence of u and its distance from the approximate solution.

The paper is organized as follows. After fixing some basic notations, Section 2 reviews the general setting of [14] for approximate NS solutions (in a reformulation suitable for our purposes, where the Reynolds number $R = 1/\nu$ and the rescaled time $t = \nu \mathfrak{t}$ are preferred to the variables ν, \mathfrak{t}). Sections 3 and 4 present the general Reynolds expansion (1.9), the control inequalities for it and some basic computational rules for the practical implementation of this approach. Section 5 applies the previous framework to the the BNW initial datum, and makes a comparison with the Galerkin approach of [14] for the same datum.

In a forthcoming paper [16] the Reynolds expansion and the related control equation will be applied to the BNW datum for larger values of N , and then employed

for other initial data of interest in the NS community, namely, the “vortices” of Taylor-Green [20] and Kida [6] .

2 Preliminaries

Throughout the paper we fix a space dimension $d \in \{2, 3, \dots\}$; in the application of section 5 we will put $d = 3$. For a, b in \mathbf{R}^d or \mathbf{C}^d we put $a \bullet b := \sum_{r=1}^d a_r b_r$ and $|a| := \sqrt{a \bullet a}$, where $\bar{}$ denotes the complex conjugation.

Let us consider the d -dimensional torus $\mathbf{T}^d := (\mathbf{R}/2\pi\mathbf{Z})^d$; the setting outlined hereafter for function spaces and NS equations on the torus is similar, but not identical to the one of [12] [14]. A main difference between these papers and the present one is that here we deal with spaces of complex valued, rather than real valued functions; some results in [12] [14] (and in other works of ours), originally formulated in a real framework, hold as well in the present, complexified setting.

In the sequel we employ the Fourier basis made of the functions ⁽²⁾

$$e_k : \mathbf{T}^d \rightarrow \mathbf{C}, \quad e_k(x) := e^{ik \bullet x} \quad (k \in \mathbf{Z}^d). \quad (2.1)$$

Let us consider the space $D'(\mathbf{T}^d, \mathbf{C}) \equiv D'$ of complex distributions on \mathbf{T}^d , i.e., the topological dual of $D := C^\infty(\mathbf{T}^d, \mathbf{C})$ ⁽³⁾. Any $v \in D'$ has a weakly convergent Fourier expansion $v = \sum_{k \in \mathbf{Z}^d} v_k e_k$, where $v_k := (2\pi)^{-d} \langle v, e_{-k} \rangle$ (the notation indicates the action of v on the function $e_{-k} \in D$).

We will often be interested in the spaces $L^p(\mathbf{T}^d, \mathbf{C}) \equiv L^p$ and, in particular, in the complex Hilbert space L^2 with the product $\langle v | w \rangle_{L^2} := \int_{\mathbf{T}^d} dx \bar{v} w = (2\pi)^d \sum_{k \in \mathbf{Z}^d} \bar{v}_k w_k$ and the corresponding norm $\| \cdot \|_{L^2}$. For all $n \in \mathbf{R}$, the n -th Sobolev space of zero mean functions on \mathbf{T}^d is

$$\begin{aligned} H_0^n(\mathbf{T}^d) &\equiv H_0^n := \left\{ v \in D' \mid \langle v \rangle = 0, \sqrt{-\Delta}^n v \in L^2 \right\} \\ &= \left\{ v \in D' \mid v_0 = 0, \sum_{k \in \mathbf{Z}^d \setminus \{0\}} |k|^{2n} |v_k|^2 < +\infty \right\} \end{aligned} \quad (2.2)$$

(in the above $\langle v \rangle \in \mathbf{C}$ is the mean of v , i.e., by definition, the action of v on the test function $1/(2\pi)^d$; moreover, $\sqrt{-\Delta}^n v := \sum_{k \in \mathbf{Z}^d \setminus \{0\}} |k|^n v_k e_k$). H_0^n is a Hilbert space with the inner product and the norm

$$\langle v | w \rangle_n := \langle \sqrt{-\Delta}^n v | \sqrt{-\Delta}^n w \rangle_{L^2} = (2\pi)^d \sum_{k \in \mathbf{Z}^d \setminus \{0\}} |k|^{2n} \bar{v}_k w_k, \quad \|v\|_n := \sqrt{\langle v | v \rangle_n}; \quad (2.3)$$

²In [12] [14], the normalization factor $(2\pi)^{-d/2}$ was included in the definition of e_k .

³The space of complex distributions carries a complex conjugation $\bar{}$, see, e.g., [12]; real distributions v on \mathbf{T}^d are characterized by the condition $\bar{v} = v$. To avoid confusion, we mention that in [12] [14] the space of complex distributions on \mathbf{T}^d is indicated with $D'_{\mathbf{C}}$, while the symbol D' is employed for the subspace of real distributions. Similar comments could be done about the Sobolev spaces mentioned later.

if $m \leq n$, then $H_0^n \subset H_0^m$.

The Laplacian and its semigroup. Let us consider the operator $\Delta := \sum_{rs=1}^d \partial_{ss} : D' \rightarrow D'$; of course $\Delta e_k = -|k|^2 e_k$ for all $k \in \mathbf{Z}^d$. For each $n \in \mathbf{R}$, Δ maps continuously H_0^{n+2} into H_0^n , with $\|\Delta v\|_n \leq \|v\|_{n+2}$ for all $v \in H_0^{n+2}$. We can define a semigroup $(e^{t\Delta})_{t \in [0, +\infty)}$ of linear operators on D' , putting

$$e^{t\Delta} : D' \rightarrow D' , \quad v \mapsto e^{t\Delta} v := \sum_{k \in \mathbf{Z}^d} e^{-|k|^2 t} v_k e_k . \quad (2.4)$$

Let $n \in \mathbf{R}$. The following holds:

$$e^{t\Delta} H_0^n \subset H_0^n , \quad \|e^{t\Delta} v\|_n \leq e^{-t} \|v\|_n \quad \text{for } t \in [0, +\infty), v \in H_0^n ; \quad (2.5)$$

$$e^{t\Delta} H_0^{n-1} \subset H_0^n ; \quad \exists \mu \in L^1((0, +\infty), \mathbf{R}) \text{ (independent of } n) \text{ such that} \quad (2.6)$$

$$\|e^{t\Delta} v\|_n \leq \mu(t) \|v\|_{n-1} \text{ for } t \in (0, +\infty), v \in H_0^{n-1}$$

(for the proof of (2.6) see, e.g., [12], that also gives an explicit expression for μ implying $\mu(t) = O(1/\sqrt{t})$ for $t \rightarrow 0^+$ and $\mu(t) = e^{-t}$ for t large). The map $(t, v) \mapsto e^{t\Delta} v$ is continuous from $[0, +\infty) \times H_0^n$ to H_0^n and from $(0, +\infty) \times H_0^{n-1}$ to H_0^n . Moreover,

$$v \in H_0^{n+2} \Rightarrow (t \mapsto e^{t\Delta} v) \in C([0, +\infty), H_0^{n+2}) \cap C^1([0, +\infty), H_0^n) , \quad (2.7)$$

$$\frac{d}{dt}(e^{t\Delta} v) = \Delta(e^{t\Delta} v) \quad \text{for } t \in [0, +\infty).$$

To go on, let

$$f \in C([0, +\infty), H_0^{n+1}) ; \quad (2.8)$$

for each $t \in [0, +\infty)$ the function $s \in (0, t) \mapsto e^{(t-s)\Delta} f(s)$ is in $L^1((0, t), H_0^{n+2})$ because, on the grounds of (2.6), $\|e^{(t-s)\Delta} f(s)\|_{n+2} \leq \mu(t-s) \|f(s)\|_{n+1}$; therefore, the definition

$$F(t) := \int_0^t ds e^{(t-s)\Delta} f(s) \quad \text{for } t \in [0, +\infty) \quad (2.9)$$

produces a function $F \in C([0, +\infty), H_0^{n+2})$. This function is also in $C^1([0, +\infty), \mathbb{H}_0^n)$, with

$$\frac{dF}{dt}(t) = \Delta F(t) + f(t) \quad \text{for } t \in [0, +\infty) . \quad (2.10)$$

Vector fields on \mathbf{T}^d . Here and in the sequel, “a vector field on \mathbf{T}^d ” means “a \mathbf{C}^d -valued distribution on \mathbf{T}^d ”. We write $\mathbb{D}'(\mathbf{T}^d) \equiv \mathbb{D}'$ for the space of such distributions; these can be identified with d -tuples $v = (v_1, \dots, v_d)$, where $v_r \in D'$ for each r . Partial derivatives, the Laplacian Δ and the operators $\sqrt{-\Delta}^n$, $e^{t\Delta}$ are defined componentwise as maps from \mathbb{D}' to \mathbb{D}' . Any $v \in \mathbb{D}'$ has a weakly convergent Fourier expansion $v = \sum_{k \in \mathbf{Z}^d} v_k e_k$, with coefficients $v_k \in \mathbf{C}^d$.

In the sequel $\mathbb{L}^p(\mathbf{T}^d) \equiv \mathbb{L}^p$ denotes the space of L^p vector fields $\mathbf{T}^d \rightarrow \mathbf{C}^d$. For each $n \in \mathbf{R}$, the n -th Sobolev space of zero mean vector fields on \mathbf{T}^d is

$$\begin{aligned} \mathbb{H}_0^n(\mathbf{T}^d) &\equiv \mathbb{H}_0^n := \left\{ v \in \mathbb{D}' \mid \langle v \rangle = 0, \sqrt{-\Delta}^n v \in \mathbb{L}^2 \right\} \\ &= \left\{ v \in \mathbb{D}' \mid v_0 = 0, \sum_{k \in \mathbf{Z}^d \setminus \{0\}} |k|^{2n} |v_k|^2 < +\infty \right\} \end{aligned} \quad (2.11)$$

(in the above, the mean $\langle v \rangle \in \mathbf{C}^d$ is defined componentwise; we note that $\mathbb{H}_0^n = \{v = (v_1, \dots, v_d) \in \mathbb{D}' \mid v_r \in H_0^n$ for each $r\}$). \mathbb{H}_0^n is a Hilbert space with the inner product and the norm

$$\langle v|w \rangle_n := \langle \sqrt{-\Delta}^n v | \sqrt{-\Delta}^n w \rangle_{L^2} = (2\pi)^d \sum_{k \in \mathbf{Z}^d \setminus \{0\}} |k|^{2n} \overline{v_k} w_k, \quad \|v\|_n := \sqrt{\langle v|v \rangle_n}. \quad (2.12)$$

Eqs. (2.5)-(2.10) and the related statements have obvious analogues, where H_0^n is replaced by \mathbb{H}_0^n for any n .

Divergence free vector fields; the Leray projection. The space of divergence free vector fields on \mathbf{T}^d is

$$\mathbb{D}'_\Sigma := \{v \in \mathbb{D}' \mid \operatorname{div} v = 0\} = \{v \in \mathbb{D}' \mid k \bullet v_k = 0 \ \forall k \in \mathbf{Z}^d\}. \quad (2.13)$$

The *Leray projection* is the linear, surjective map

$$\mathfrak{L} : \mathbb{D}' \rightarrow \mathbb{D}'_\Sigma, \quad v \mapsto \mathfrak{L}v := \sum_{k \in \mathbf{Z}^d} (\mathfrak{L}_k v_k) e_k; \quad (2.14)$$

here \mathfrak{L}_k is the orthogonal projection of \mathbf{C}^d onto the orthogonal complement of k , i.e.,

$$\mathfrak{L}_0 c = c, \quad \mathfrak{L}_k c = c - \frac{k \bullet c}{|k|^2} k \quad \text{for } c \in \mathbf{C}^d, k \in \mathbf{Z}^d \setminus \{0\}. \quad (2.15)$$

For each $n \in \mathbf{R}$, the n -th Sobolev space of zero mean, divergence free vector fields is

$$\mathbb{H}_{\Sigma 0}^n := \mathbb{D}'_\Sigma \cap \mathbb{H}_0^n; \quad (2.16)$$

this is a closed subspace of \mathbb{H}_0^n , and thus becomes a Hilbert space with the restriction of the inner product $\langle \cdot | \cdot \rangle_n$. One has

$$\mathfrak{L} \mathbb{H}_{\Sigma 0}^n = \mathbb{H}_{\Sigma 0}^n, \quad \|\mathfrak{L}v\|_n \leq \|v\|_n \quad \text{for } n \in \mathbf{R}, v \in \mathbb{H}_0^n. \quad (2.17)$$

The spaces (2.16) are the basis of our treatment of the NS equations; again, we have analogues of Eqs. (2.5)-(2.10) and of the related statements, where H_0^n is replaced by $\mathbb{H}_{\Sigma 0}^n$ for any n .

The NS bilinear map. Consider two vector fields v, w on \mathbf{T}^d such that $v \in \mathbb{L}^2$ and $\partial_r w \in \mathbb{L}^2$ for $r = 1, \dots, d$; then we have a well defined vector field $v \bullet \partial w \in \mathbb{L}^1$ of components $(v \bullet \partial w)_r := \sum_{s=1}^d v_s \partial_s w_r$ (which has mean zero if $\operatorname{div} v = 0$); we can apply to this the Leray projection \mathfrak{L} and form the (divergence free) vector field

$$\mathcal{P}(v, w) := -\mathfrak{L}(v \bullet \partial w) . \quad (2.18)$$

The bilinear map \mathcal{P} : $(v, w) \mapsto \mathcal{P}(v, w)$, which is a main character of the incompressible NS equations, is known to possess the following properties.

- (i) For each $n > d/2$, \mathcal{P} is continuous from $\mathbb{H}_{\Sigma_0}^n \times \mathbb{H}_{\Sigma_0}^{n+1}$ to $\mathbb{H}_{\Sigma_0}^n$; so, there is a constant $K_{nd} \equiv K_n$ such that

$$\|\mathcal{P}(v, w)\|_n \leq K_n \|v\|_n \|w\|_{n+1} \quad \text{for } v \in \mathbb{H}_{\Sigma_0}^n, w \in \mathbb{H}_{\Sigma_0}^{n+1} . \quad (2.19)$$

- (ii) For each $n > d/2 + 1$, there is a constant $G_{nd} \equiv G_n$ such that

$$|\langle \mathcal{P}(v, w) | w \rangle_n| \leq G_n \|v\|_n \|w\|_n^2 \quad \text{for } v \in \mathbb{H}_{\Sigma_0}^n, w \in \mathbb{H}_{\Sigma_0}^{n+1} \quad (2.20)$$

(this result is due to Kato, see [4]).

From here to the end of the paper, K_n and G_n are constants fulfilling the previous inequalities (and not necessarily sharp). From [13] [15] we know that we can take

$$K_3 = 0.323 , \quad G_3 = 0.438 \quad \text{if } d = 3 ; \quad (2.21)$$

these values will be useful in the sequel.

The NS Cauchy problem. From here to the end of the paper, we fix a Sobolev order

$$n \in \left(\frac{d}{2} + 1, +\infty \right) . \quad (2.22)$$

Let us choose a Reynolds number $R \in [0, +\infty)$ and an initial datum

$$u_* \in \mathbb{H}_{\Sigma_0}^{n+2} . \quad (2.23)$$

2.1 Definition. *The (incompressible) NS Cauchy problem with Reynolds number R and initial datum u_* is the following:*

$$\text{Find } u \in C([0, T], \mathbb{H}_{\Sigma_0}^{n+2}) \cap C^1([0, T], \mathbb{H}_{\Sigma_0}^n) \quad \text{such that} \quad (2.24)$$

$$\frac{du}{dt} = \Delta u + R \mathcal{P}(u, u) , \quad u(0) = u_*$$

(with $T \in (0, +\infty]$, depending on u).

It is known [5] that the above Cauchy problem has a unique maximal (i.e., not extendable) solution; any solution is a restriction of the maximal one. ⁽⁴⁾

Approximate solutions of the NS Cauchy problem. Let us briefly rephrase the results of [14], with the notations of the present paper (this is advisable, since in [14] we wrote the NS equations in the form (1.1), rather than (1.4)). We consider again the Cauchy problem (2.24), for given n, R, u_* as above; the definitions and the theorem that follow are reported from [14], with obvious adaptations.

2.2 Definition. An approximate solution of the problem (2.24) is any map $u_a \in C([0, T_a], \mathbb{H}_{\Sigma_0}^{n+2}) \cap C^1([0, T_a], \mathbb{H}_{\Sigma_0}^n)$ (with $T_a \in (0, +\infty]$). Given such a function, we stipulate (i) (ii).

(i) The differential error of u_a is

$$\frac{du_a}{dt} - \Delta u_a - R \mathcal{P}(u_a, u_a) \in C([0, T_a], \mathbb{H}_{\Sigma_0}^n) ; \quad (2.25)$$

the datum error is

$$u_a(0) - u_* \in \mathbb{H}_{\Sigma_0}^{n+2} . \quad (2.26)$$

(ii) Let $m \in \mathbf{R}, m \leq n$. A differential error estimator of order m for u_a is a function $\epsilon_m \in C([0, T_a], [0, +\infty))$ such that

$$\|(\frac{du_a}{dt} - \Delta u_a - R \mathcal{P}(u_a, u_a))(t)\|_m \leq \epsilon_m(t) \quad \text{for } t \in [0, T_a] . \quad (2.27)$$

Let $m \in \mathbf{R}, m \leq n+2$. A datum error estimator of order m for u_a is a real number $\delta_m \in [0, +\infty)$ such that

$$\|u_a(0) - u_*\|_m \leq \delta_m ; \quad (2.28)$$

a growth estimator of order m for u_a is a function $\mathcal{D}_m \in C([0, T_a], [0, +\infty))$ such that

$$\|u_a(t)\|_m \leq \mathcal{D}_m(t) \quad \text{for } t \in [0, T_a] . \quad (2.29)$$

In particular the function $\epsilon_m(t) := \|(du_a/dt - \Delta u_a - R \mathcal{P}(u_a, u_a))(t)\|_m$, the number $\delta_m := \|u_a(0) - u_*\|_m$ and the function $\mathcal{D}_m(t) := \|u_a(t)\|_m$ will be called the tautological estimators of order m for the differential error, the datum error and the growth of u_a .

From here to the end of the section we consider an approximate solution u_a of the problem (2.24), with domain $[0, T_a]$; this is assumed to possess differential, datum error and growth estimators of orders n or $n+1$, indicated with $\epsilon_n, \delta_n, \mathcal{D}_n, \mathcal{D}_{n+1}$.

⁴It is known as well that, if u solves (2.24), the function $(x, t) \mapsto u(x, t)$ is smooth on $\mathbf{T}^d \times (0, T)$ (see, e.g., Theorem 15.2 (A) of [7]); one could give stronger regularity results with suitable assumptions on u_* . In the sequel we will not be interested in such regularity matters.

2.3 Definition. Let $\mathcal{R}_n \in C([0, T_c], [0, +\infty))$, with $T_c \in (0, T_a]$. This function is said to fulfill the control inequalities if

$$\frac{d^+ \mathcal{R}_n}{dt} \geq -\mathcal{R}_n + R(G_n \mathcal{D}_n + K_n \mathcal{D}_{n+1}) \mathcal{R}_n + R G_n \mathcal{R}_n^2 + \epsilon_n \quad \text{in } [0, T_c), \quad (2.30)$$

$$\mathcal{R}_n(0) \geq \delta_n. \quad (2.31)$$

In the above d^+/dt indicates the right, upper Dini derivative: so, for all $t \in [0, T_c)$, $(d^+ \mathcal{R}_n/dt)(t) := \limsup_{h \rightarrow 0^+} [\mathcal{R}_n(t+h) - \mathcal{R}_n(t)]/h$.

2.4 Proposition. Assume there is a function $\mathcal{R}_n \in C([0, T_c], [0, +\infty))$ fulfilling the control inequalities; consider the maximal solution u of the NS Cauchy problem (2.24), and denote its domain with $[0, T)$. Then

$$T \geq T_c, \quad (2.32)$$

$$\|u(t) - u_a(t)\|_n \leq \mathcal{R}_n(t) \quad \text{for } t \in [0, T_c). \quad (2.33)$$

For completeness we mention that any function $\mathcal{R}_n \in C([0, T_c], \mathbf{R})$ fulfilling the inequalities (2.30) (2.31) is automatically nonnegative ⁽⁵⁾.

Paper [14] presents some applications of the previous proposition, where u_a is constructed by the Galerkin method. (For completeness we mention that the framework of [14] also covers the case of the Euler equations, i.e., the limit case $\nu \rightarrow 0$ of (1.1) which is formally equivalent to $R \rightarrow +\infty$; some applications to the Euler equations have been considered both in [14] and in [9].)

In the sequel of this paper we present an application of Proposition 2.4, choosing for u_a a polynomial in R (see Eq.(1.9)). In the next two sections we develop this approach in general terms, giving the error estimators for an approximate solution of this kind; in the final section we apply this procedure choosing for u_* the BNW initial datum.

3 Reynolds number expansions as approximate NS solutions

Let us recall that $n \in (d/2 + 1, +\infty)$, and consider the NS Cauchy problem (2.24) with $R \in [0, +\infty)$ and datum $u_* \in \mathbb{H}_{\Sigma_0}^{n+2}$. Let us choose an order $N \in \{0, 1, 2, \dots\}$

⁵In fact the zero function fulfills relations analogous to (2.30) (2.31), with \geq replaced by \leq ; by standard comparison results, this implies $\mathcal{R}_n(t) \geq 0$ for all $t \in [0, T_c)$. The comparison results required to prove the last statement have been presented in [14] (see Appendix A of the cited paper and references therein).

and consider as an approximate solution for (2.24) a polynomial of degree N in R , of the form

$$u^N : [0, +\infty) \rightarrow \mathbb{H}_{\Sigma_0}^{n+2}, \quad t \mapsto u^N(t) := \sum_{j=0}^N R^j u_j(t), \quad (3.1)$$

$$u_j \in C([0, +\infty) \mathbb{H}_{\Sigma_0}^{n+2}) \cap C^1([0, +\infty), \mathbb{H}_{\Sigma_0}^n) \quad \text{for } j = 0, \dots, N;$$

the functions u_j herein are to be determined.

3.1 Proposition. (i) Let u^N be as in (3.1). The datum and differential errors of u^N are

$$u^N(0) - u_* = (u_0(0) - u_*) + \sum_{j=1}^N R^j u_j(0); \quad (3.2)$$

$$\frac{du^N}{dt} - \Delta u^N - \mathcal{P}(u^N, u^N) \quad (3.3)$$

$$= \left(\frac{du_0}{dt} - \Delta u_0 \right) + \sum_{j=1}^N R^j \left[\frac{du_j}{dt} - \Delta u_j - \sum_{\ell=0}^{j-1} \mathcal{P}(u_\ell, u_{j-\ell-1}) \right] - \sum_{j=N+1}^{2N+1} R^j \sum_{\ell=j-N-1}^N \mathcal{P}(u_\ell, u_{j-\ell-1}).$$

(ii) One can define recursively a family of functions $u_j \in C([0, +\infty), \mathbb{H}_{\Sigma_0}^{n+2}) \cap C^1([0, +\infty), \mathbb{H}_{\Sigma_0}^n)$ setting

$$u_0(t) := e^{t\Delta} u_* \quad \text{for } t \in [0, +\infty), \quad (3.4)$$

$$u_j(t) := \sum_{\ell=0}^{j-1} \int_0^t ds e^{(t-s)\Delta} \mathcal{P}(u_\ell(s), u_{j-\ell-1}(s)) \quad \text{for } t \in [0, +\infty), j = 1, \dots, N. \quad (3.5)$$

With this choice of u_0, \dots, u_N the coefficients of R^0, R^1, \dots, R^N in Eqs. (3.2) and (3.3) vanish, so that

$$u^N(0) - u_* = 0; \quad (3.6)$$

$$\frac{du^N}{dt} - \Delta u^N - \mathcal{P}(u^N, u^N) = - \sum_{j=N+1}^{2N+1} R^j \sum_{\ell=j-N-1}^N \mathcal{P}(u_\ell, u_{j-\ell-1}). \quad (3.7)$$

The second equation implies

$$\left\| \left(\frac{du^N}{dt} - \Delta u^N - R \mathcal{P}(u^N, u^N) \right)(t) \right\|_n \quad (3.8)$$

$$\leq K_n \sum_{j=N+1}^{2N+1} R^j \sum_{\ell=j-N-1}^N \|u_\ell(t)\|_n \|u_{j-\ell-1}(t)\|_{n+1} \quad \text{for all } t \in [0, +\infty).$$

Proof. (i) Eq. (3.2) is obvious. Let us prove Eq. (3.3); to this purpose, we note that

$$\begin{aligned}
\frac{du^N}{dt} - \Delta u^N - R\mathcal{P}(u^N, u^N) &= \left(\frac{d}{dt} - \Delta\right) \left(\sum_{j=0}^N R^j u_j\right) - R\mathcal{P}\left(\sum_{\ell=0}^N R^\ell u_\ell, \sum_{h=0}^N R^h u_h\right) \\
&= \sum_{j=0}^N R^j \left(\frac{du_j}{dt} - \Delta u_j\right) - \sum_{\ell, h=0}^N R^{\ell+h+1} \mathcal{P}(u_\ell, u_h) \\
&= \left(\frac{du_0}{dt} - \Delta u_0\right) + \sum_{j=1}^N R^j \left(\frac{du_j}{dt} - \Delta u_j\right) - \sum_{j=1}^{2N+1} R^j \sum_{(\ell, h) \in I_{Nj}} \mathcal{P}(u_\ell, u_h) , \\
I_{Nj} &:= \{(\ell, h) \in \{0, \dots, N\}^2 \mid \ell + h + 1 = j\} .
\end{aligned}$$

One easily checks that

$$j \in \{1, \dots, N\} \Rightarrow I_{Nj} = \{(\ell, j - \ell - 1) \mid \ell \in \{0, \dots, j - 1\}\} ,$$

$$j \in \{N + 1, \dots, 2N + 1\} \Rightarrow I_{Nj} = \{(\ell, j - \ell - 1) \mid \ell \in \{j - N - 1, \dots, N\}\} ;$$

this readily yields the thesis (3.3).

(ii) First of all (by an obvious vector analogue of (2.7)), Eq. (3.4) defines a function $u_0 \in C([0, +\infty), \mathbb{H}_{\Sigma_0}^{n+2}) \cap C^1([0, +\infty), \mathbb{H}_{\Sigma_0}^n)$ such that

$$u_0(0) = u_* , \quad \frac{du_0}{dt} = \Delta u_0 ; \quad (3.9)$$

to go on, let us prove by recurrence over j that the functions u_j are well defined and belong to $C([0, +\infty), \mathbb{H}_{\Sigma_0}^{n+2}) \cap C^1([0, +\infty), \mathbb{H}_{\Sigma_0}^n)$, for all $j \leq N$. For $j = 0$, the thesis has been already established. Now, let $j \in \{1, \dots, N\}$ and assume the thesis to hold up to the order $j - 1$; then the functions $t \in [0, +\infty) \mapsto \mathcal{P}(u_\ell(t), u_{j-\ell-1}(t))$ ($\ell = 0, \dots, j - 1$) are in $C([0, +\infty), \mathbb{H}_{\Sigma_0}^{n+1})$ due to the properties of \mathcal{P} . By the obvious vector analogues of the considerations accompanying Eqs. (2.9) (2.10), with f replaced by $\mathcal{P}(u_\ell, u_{j-\ell-1})$, we see that the functions $t \mapsto \int_0^t ds e^{(t-s)\Delta} \mathcal{P}(u_\ell(s), u_{j-\ell-1}(s))$ belong to $C([0, +\infty), \mathbb{H}_{\Sigma_0}^{n+2}) \cap C^1([0, +\infty), \mathbb{H}_{\Sigma_0}^n)$; the same can be said of u_j , which is the sum over ℓ of these functions.

Let us prove that, with the present choice of the functions u_j ($j = 0, \dots, N$), the coefficients of R^0, \dots, R^N in Eqs. (3.2) (3.3) are zero. First of all, the definitions (3.4) (3.5) imply $u_0(0) = u_*$ and $u_j(0) = 0$ for $j = 1, \dots, N$; so, the coefficients of R^0, \dots, R^N in (3.2) are zero. To go on, we note that Eq. (3.9) indicates the vanishing of the coefficient of R^0 in (3.3). For $j = 1, \dots, N$, using Eq. (2.10) (in a vector analogue, with f replaced by $\sum_\ell \mathcal{P}(u_\ell, u_{j-\ell-1})$), we see that

$$\frac{du_j}{dt}(t) = \Delta u_j(t) + \sum_{\ell=0}^{j-1} \mathcal{P}(u_\ell(t), u_{j-\ell-1}(t)) ; \quad (3.10)$$

this proves the vanishing of the coefficient of R^j in Eq. (3.3).

Now Eqs. (3.6) (3.7) are obvious. Eq. (3.7) and the inequality $\|\mathcal{P}(u_\ell(t), u_{j-\ell-1}(t))\|_n \leq K_n \|u_\ell(t)\|_n \|u_{j-\ell-1}(t)\|_{n+1}$ gives immediately Eq. (3.8). \square

Let $N \in \{0, 1, 2, \dots\}$; define u^N and u_j ($j = 0, \dots, N$) via Eqs. (3.1) (3.4) (3.5). We regard u^N as an approximate solution of the Cauchy problem (2.24), of domain $[0, +\infty)$; this has the tautological datum error and growth estimators

$$\delta_n := 0, \quad \mathcal{D}_n(t) := \|u^N(t)\|_n, \quad \mathcal{D}_{n+1}(t) := \|u^N(t)\|_{n+1}. \quad (3.11)$$

The differential error $du^N/t - \Delta u^N - \mathcal{P}(u^N, u^N)$ can be expressed via Eq. (3.7); we can use for it the tautological estimator

$$\epsilon_n(t) := \left\| \left(\frac{du^N}{dt} - \Delta u^N - R\mathcal{P}(u^N, u^N) \right)(t) \right\|_n \quad (3.12)$$

or the rougher estimator indicated by (3.8), i.e., the function

$$\epsilon_n(t) := K_n \sum_{j=N+1}^{2N+1} R^j \sum_{\ell=j-N+1}^N \|u_\ell(t)\|_n \|u_{j-\ell-1}(t)\|_{n+1}. \quad (3.13)$$

Now, using Proposition 2.4 with $u_a = u^N$ and the above estimators we obtain the following.

3.2 Corollary. *Let $N \in \{0, 1, 2, \dots\}$; define $\delta_n, \mathcal{D}_n, \mathcal{D}_{n+1}$ via Eq. (3.11) and ϵ_n via Eq. (3.12) or (3.13), for $t \in [0, +\infty)$. Suppose there is a function $\mathcal{R}_n \in C([0, T_c), [0, +\infty))$, with $T_c \in (0, +\infty]$, fulfilling the control inequalities (2.30) (2.31). Consider the maximal solution u of the NS Cauchy problem (2.24), of domain $[0, T)$; then*

$$T \geq T_c, \quad \|u(t) - u^N(t)\|_n \leq \mathcal{R}_n(t) \quad \text{for } t \in [0, T_c). \quad (3.14)$$

In particular, the control inequalities (2.30) (2.31) are satisfied by any function $\mathcal{R}_n \in C^1([0, T_c), [0, +\infty))$ fulfilling the control Cauchy problem

$$\frac{d\mathcal{R}_n}{dt} = -\mathcal{R}_n + R(G_n \mathcal{D}_n + K_n \mathcal{D}_{n+1})\mathcal{R}_n + R G_n \mathcal{R}_n^2 + \epsilon_n \text{ on } [0, T_c), \quad \mathcal{R}_n(0) = 0. \quad (3.15)$$

3.3 Remarks. (i) Even though rougher than the estimator (3.12), the estimator (3.13) is interesting because its computation is less expensive; this is a relevant fact, especially in applications with a large N .

(ii) Let us consider the estimator (3.12), writing via (3.7) the error therein; alternatively, let us use the estimator (3.13). In both cases, it is natural to write

$$\epsilon_n(t) = R^{N+1} \tilde{\epsilon}_n(t) \quad \text{for } t \in [0, +\infty), \quad (3.16)$$

where $\tilde{\epsilon}(t)$ is a suitable function; this is a polynomial of degree N in R in the case (3.13), and the square root of a polynomial of degree $2N$ in R in the case (3.12). Consequently, the solution of the Cauchy problem (3.15) can be written as

$$\mathcal{R}_n(t) = R^{N+1} \tilde{\mathcal{R}}_n(t) , \quad (3.17)$$

where $\tilde{\mathcal{R}}_n \in C^1([0, T_c), [0, +\infty))$ is such that

$$\frac{d\tilde{\mathcal{R}}_n}{dt} = -\tilde{\mathcal{R}}_n + R(G_n \mathcal{D}_n + K_n \mathcal{D}_{n+1}) \tilde{\mathcal{R}}_n + R^{N+2} G_n \tilde{\mathcal{R}}_n^2 + \tilde{\epsilon}_n , \quad \tilde{\mathcal{R}}_n(0) = 0. \quad (3.18)$$

(iii) Eq. (3.14) has a number of obvious implications; let us give two examples.

(iii₁) The inequalities $\|u^N(t)\|_n - \|u(t) - u^N(t)\|_n \leq \|u(t)\|_n \leq \|u^N(t)\|_n + \|u(t) - u^N(t)\|_n$, the definition of $\mathcal{D}_n(t)$ in (3.11) and Eq. (3.14) for \mathcal{R}_n imply

$$\mathcal{D}_n(t) - \mathcal{R}_n(t) \leq \|u(t)\|_n \leq \mathcal{D}_n(t) + \mathcal{R}_n(t) \quad \text{for } t \in [0, T_c) . \quad (3.19)$$

(iii₂) For any $v \in \mathbb{H}_0^n$, the equation $\|v\|_n^2 = (2\pi)^d \sum_{k \in \mathbf{Z}^d \setminus \{0\}} |k|^{2n} |v_k|^2$ implies $(2\pi)^{d/2} |v_k| \leq \|v\|_n / |k|^n$ for all k . This inequality with $v = u(t) - u^N(t)$ and (3.14) give

$$(2\pi)^{d/2} |u_k(t) - u_k^N(t)| \leq \frac{\mathcal{R}_n(t)}{|k|^n} \quad \text{for } k \in \mathbf{Z}^d \setminus \{0\} \text{ and } t \in [0, T_c) . \quad (3.20)$$

4 Implementing the recursion relations (3.4) (3.5)

The recursion relations mentioned in the title are the main characters of Proposition 3.1; their building blocks are, essentially, the linear maps \mathcal{U}, \mathcal{K} defined as follows:

$$\mathcal{U} : H_0^{n+2} \rightarrow C([0, +\infty), H_0^{n+2}) \cap C^1([0, +\infty), H_0^n), \quad z \mapsto \mathcal{U}z \quad (4.1)$$

$$\text{where } \mathcal{U}z : [0, +\infty) \rightarrow H_0^{n+2} , \quad t \mapsto (\mathcal{U}z)(t) := e^{t\Delta} z ;$$

$$\mathcal{K} : C([0, +\infty), H_0^{n+1}) \rightarrow C([0, +\infty), H_0^{n+2}) \cap C^1([0, +\infty), H_0^n) , \quad f \mapsto \mathcal{K}f \quad (4.2)$$

$$\text{where } \mathcal{K}f : [0, +\infty) \rightarrow H_0^{n+2} , \quad t \mapsto (\mathcal{K}f)(t) := \int_0^t ds e^{(t-s)\Delta} f(s)$$

(as for the domains and codomains of the above maps, recall Eqs. (2.7) (2.10) and the related comments). These maps have vector analogues, denoted for simplicity with the same letters,

$$\mathcal{U} : \mathbb{H}_0^{n+2} \rightarrow C([0, +\infty), \mathbb{H}_0^{n+2}) \cap C^1([0, +\infty), \mathbb{H}_0^n), \quad z \mapsto \mathcal{U}z , \quad (4.3)$$

$$\mathcal{K} : C([0, +\infty), \mathbb{H}_0^{n+1}) \rightarrow C([0, +\infty), \mathbb{H}_0^{n+2}) \cap C^1([0, +\infty), \mathbb{H}_0^n) , \quad f \mapsto \mathcal{K}f ; \quad (4.4)$$

these are defined rephrasing Eqs. (4.1) (4.2). In the sequel, it will always be clear from the context whether the symbols \mathcal{U}, \mathcal{K} refer to the maps (4.1) (4.2) or to the maps (4.3) (4.4). For $z = (z_1, \dots, z_d) \in \mathbb{H}_{\Sigma_0}^{n+2}$ and $f = (f_1, \dots, f_d) \in C([0, +\infty), \mathbb{H}_0^{n+1})$ we have

$$\mathcal{U}z = (\mathcal{U}z_1, \dots, \mathcal{U}z_d) , \quad \mathcal{K}f = (\mathcal{K}f_1, \dots, \mathcal{K}f_d) . \quad (4.5)$$

With the above notations, for any $u_* \in \mathbb{H}_{\Sigma_0}^{n+2}$ the recursion relations (3.4) (3.5) can be written as

$$u_0 := \mathcal{U}u_* , \quad (4.6)$$

$$u_j := \sum_{\ell=0}^{j-1} \mathcal{K}\mathcal{P}(u_\ell, u_{j-\ell-1}) \quad (j = 1, \dots, N) . \quad (4.7)$$

Hereafter we give a set of elementary computational rules about the maps \mathcal{U}, \mathcal{K} ; these allow a straightforward computation of the functions u_0, u_1, \dots, u_N in Eqs. (4.6) (4.7), at least in the case when u_* is a Fourier polynomial (see the remark at the end of this section). Our computational rules involve the functions $e_k \in D$, $e_k(x) := e^{ik \bullet x}$ ($k \in \mathbf{Z}^d$) and

$$B_{a,b} : [0, +\infty) \rightarrow \mathbf{R} , \quad t \mapsto B_{a,b}(t) := t^a e^{-bt} \quad (a, b \in \mathbf{N}) , \quad (4.8)$$

as well as the products

$$B_{a,b} e_k : [0, +\infty) \rightarrow D , \quad t \mapsto t^a e^{-bt} e_k ; \quad (4.9)$$

they can be summarized in the following statement.

4.1 Lemma. *Let $a, a', b, b' \in \mathbf{N}$, $k, k' \in \mathbf{Z}^d$ and $s \in \{1, \dots, d\}$. Then*

$$\partial_s(B_{a,b} e_k) = ik_s B_{a,b} e_k ; \quad (4.10)$$

$$(B_{a,b} e_k)(B_{a',b'} e_{k'}) = B_{a+a', b+b'} e_{k+k'} . \quad (4.11)$$

In addition, let $k \neq 0$ (so that the functions below are in the domains of \mathcal{U} or \mathcal{K}). Then

$$\mathcal{U}e_k = B_{0,|k|^2} e_k ; \quad (4.12)$$

$$\mathcal{K}(B_{a,b} e_k) = a! \left(\frac{B_{0,|k|^2}}{(b - |k|^2)^{a+1}} - \sum_{\ell=0}^a \frac{B_{\ell,b}}{(b - |k|)^{a+1-\ell} \ell!} \right) e_k \quad \text{if } b \neq |k|^2; \quad (4.13)$$

$$\mathcal{K}(B_{a,|k|^2} e_k) = \frac{B_{a+1,|k|^2}}{a+1} e_k . \quad (4.14)$$

Proof. Eqs. (4.10) (4.11) are obvious consequences of the definitions of e_k and $B_{a,b}$. Eq. (4.12) is just a reformulation of the relation $e^{t\Delta}e_k = e^{-|k|^2 t}e_k$.

In order to derive Eqs. (4.13) (4.14), we note that (both for $b \neq |k|^2$ and for $b = |k|^2$),

$$\begin{aligned}\mathcal{K}(B_{a,b}e_k)(t) &= \int_0^t ds e^{(t-s)\Delta} (s^a e^{-bs} e_k) \\ &= \int_0^t ds s^a e^{-bs} e^{-|k|^2(t-s)} e_k = e^{-|k|^2 t} \left(\int_0^t ds s^a e^{-(b-|k|^2)s} \right) e_k.\end{aligned}\tag{4.15}$$

Let $b \neq |k|^2$. Then, a change of variables $\sigma = (b - |k|^2)s$ gives $\int_0^t ds s^a e^{-(b-|k|^2)s} = 1/(b - |k|^2)^{a+1} \int_0^{(b-|k|^2)t} d\sigma \sigma^a e^{-\sigma}$; thus

$$\mathcal{K}(B_{a,b}e_k)(t) = \frac{e^{-|k|^2 t}}{(b - |k|^2)^{a+1}} \gamma(a+1, (b - |k|^2)t) e_k, \tag{4.16}$$

where we have introduced the incomplete Gamma function

$$\gamma(\alpha, y) := \int_0^y d\sigma \sigma^{\alpha-1} e^{-\sigma} \quad \text{for } y \in \mathbf{R}, \alpha \in \{1, 2, 3, \dots\}. \tag{4.17}$$

It is known that

$$\gamma(a+1, y) = a! \left(1 - e^{-y} \sum_{\ell=0}^a \frac{y^\ell}{\ell!} \right) \tag{4.18}$$

(see, e.g., [17], page 177, Eqs. (8.4.7) and (8.4.11)). Inserting this result into (4.16) we obtain that, for $b \neq |k|^2$,

$$\mathcal{K}(B_{a,b}e_k)(t) = a! \left(\frac{e^{-|k|^2 t}}{(b - |k|^2)^{a+1}} - \sum_{\ell=0}^a \frac{t^\ell e^{-bt}}{(b - |k|^2)^{a+1-\ell} \ell!} \right) e_k; \tag{4.19}$$

this proves Eq. (4.13). Finally, in the case $b = |k|^2$ we obtain from (4.15) that

$$\mathcal{K}(B_{a,|k|^2}e_k)(t) = e^{-|k|^2 t} \left(\int_0^t ds s^a \right) e_k = \frac{t^{a+1} e^{-|k|^2 t}}{a+1} e_k; \tag{4.20}$$

this proves Eq. (4.14). □

4.2 Remark. Let us return to the recursion rules (4.6) (4.7), assuming that the initial datum u_* is a Fourier polynomial: by this we mean that u_* has finitely many non zero Fourier coefficients. In this case, due to Eqs. (4.6) and (4.12) all components of u_0 are linear combinations of finitely many functions of the form

$B_{0,|k|^2} e_k$. Using Eq. (4.7) with the results of the previous lemma we see that, for $j = 1, \dots, N$, each component of u_j is a finite linear combination of functions of the form $B_{a,b} e_k$. We note that each term $\mathcal{P}(u_\ell, u_{j-\ell-1})$ in Eq. (4.7) can be calculated as follows: first of all, one computes each component of $u_\ell \bullet \partial u_{j-\ell-1}$ using elementary considerations of bilinearity, together with Eqs. (4.10) (4.11); next, one obtains $\mathcal{P}(u_\ell, u_{j-\ell-1}) = -\mathfrak{L}(u_\ell \bullet \partial u_{j-\ell-1})$ using the expression (2.14)(2.15) for the Leray projection.

5 An application with the Behr-Nečas-Wu (BNW) initial datum

Throughout this section we work with

$$d = 3, \quad n = 3 \quad (5.1)$$

and any Reynolds number $R \in [0, +\infty)$. The Cauchy problem (2.24) takes the form

$$\text{Find } u \in C([0, T), \mathbb{H}_{\Sigma_0}^5) \cap C^1([0, T), \mathbb{H}_{\Sigma_0}^3) \text{ such that} \quad (5.2)$$

$$\frac{du}{dt} = \Delta u + R \mathcal{P}(u, u), \quad u(0) = u_*.$$

The initial datum u_* in $\mathbb{H}_{\Sigma_0}^5$ (in fact, in $\mathbb{H}_{\Sigma_0}^m$ for any real m) is chosen as follows:

$$u_*(x_1, x_2, x_3) := 2(\cos(x_1 + x_2) + \cos(x_1 + x_3), \quad (5.3)$$

$$-\cos(x_1 + x_2) + \cos(x_2 + x_3), -\cos(x_1 + x_3) - \cos(x_2 + x_3)) .$$

This is a Fourier polynomial; indeed

$$u_* = \sum_{k=\pm a, \pm b, \pm c} u_{*k} e_k, \quad (5.4)$$

$$a := (1, 1, 0), \quad b := (1, 0, 1), \quad c := (0, 1, 1);$$

$$u_{*,\pm a} := (1, -1, 0), \quad u_{*,\pm b} := (1, 0, -1), \quad u_{*,\pm c} := (0, 1, -1).$$

This initial datum has been introduced by Behr, Nečas and Wu in [1]; these authors have considered the datum (5.3) (5.4) as the origin of a possible blow-up for the Euler equations (i.e., for the zero viscosity limit of (1.1)).

We have disputed the BNW blow-up conjecture in [8]; in the present paper, independently of any opinion on the validity of this conjecture, we consider the datum (5.3) (5.4) in presence of viscosity and analyze it by the Reynolds expansion method of sections 3 and 4. In the final subsection we compare the outcomes of this approach with the results on the BNW datum obtained for the viscous case in

[14], where a Galerkin approximate solution was employed in place of the Reynolds expansion.

Setting up the Reynolds expansion. The expansion of order N relies on the function $u^N := \sum_{j=0}^N R^j u_j : [0, +\infty) \rightarrow \mathbb{H}_{\Sigma_0}^5$ where u_0, \dots, u_N are computed via the recursion rules (4.6)–(4.7), starting from the BNW datum (5.4); after finding u^N , one computes the related growth and error estimators $\mathcal{D}_3, \mathcal{D}_4, \epsilon_3$ and sets up the control Cauchy problem

$$\frac{d\mathcal{R}_3}{dt} = -\mathcal{R}_3 + R(G_3\mathcal{D}_3 + K_3\mathcal{D}_4)\mathcal{R}_3 + R G_3\mathcal{R}_3^2 + \epsilon_3 \text{ on } [0, T_c), \quad \mathcal{R}_3(0) = 0, \quad (5.5)$$

with K_3 and G_3 as in (2.21). In the sequel

$$\mathcal{R}_3 : [0, T_c) \rightarrow [0, +\infty) \quad (5.6)$$

always denotes the maximal solution of this problem ⁽⁶⁾.

The above conceptual scheme has been implemented in the cases $N = 1, 2, 5$, on which we report in the sequel. The case $N = 1$ is useful to illustrate in full detail the method; the other ones give more accurate results, at the price of more expensive computations. In all cases, for the practical computation of u_0, u_1, \dots and of the estimators $\mathcal{D}_3, \mathcal{D}_4, \epsilon_3$, we have employed MATHEMATICA in the symbolic mode. Then the control Cauchy problem has been solved numerically (using again MATHEMATICA) for several sample values of R ; these numerical calculations are very reliable, since they concern a simple one-dimensional ODE. The outcomes of such computations give evidence for the following picture.

- (i) There is a critical Reynolds number R_* (depending on N) such that $T_c = +\infty$ for $0 \leq R \leq R_*$, and $T_c < +\infty$ for $R > R_*$. Moreover, for $0 \leq R \leq R_*$ one has $\mathcal{R}_3(t) \rightarrow 0^+$ for $t \rightarrow +\infty$, while for $R > R_*$ one has $\mathcal{R}_3(t) \rightarrow +\infty$ for $t \rightarrow T_c^-$.

On the grounds of our general setting, the results (i) on the control Cauchy problem yield the following conclusion.

- (ii) Let us consider the maximal solution u of the NS Cauchy problem (5.2) and the Reynolds expansion (for $N = 1, 2$ or 5), with its critical number R_* . For $0 \leq R \leq R_*$, u is global and $\|u(t) - u^N(t)\|_3 \leq \mathcal{R}_3(t)$ for all $t \in [0, +\infty)$. For $R > R_*$, we can only grant that the domain of u contains the interval $[0, T_c)$, and that $\|u(t) - u^N(t)\|_3 \leq \mathcal{R}_3(t)$ for all $t \in [0, T_c)$.

⁶More precisely: one considers the real valued C^1 functions fulfilling (5.5), which are automatically nonnegative (see the comment immediately after Proposition 2.4); the maximal C^1 solution (i.e., the solution with the largest domain) is the function in (5.6).

In the sequel we give more detailed information about each one of the cases $N = 1, 2, 5$. As expected, when N is increased the critical value R_* increases as well. Our tests also indicate that, for a given R , when N increases the same happens of T_c ; on the contrary \mathcal{R}_3 becomes smaller, thus giving a more stringent estimate on $\|u(t) - u^N(t)\|_3$.

Case $N = 1$. This relies on

$$u^1 := u_0 + Ru_1 : [0, +\infty) \rightarrow \mathbb{H}_{\Sigma_0}^5 . \quad (5.7)$$

The expressions of $u_j = (u_{j,1}, u_{j,2}, u_{j,3})$ for $j = 0, 1$ are as follows (with $B_{a,b}(t) := t^a e^{-bt}$, as in the previous section):

$$\begin{aligned} u_{0,1} &= B_{0,2}(e_{(-1,-1,0)} + e_{(-1,0,-1)} + e_{(1,0,1)} + e_{(1,1,0)}) , \\ u_{0,2} &= B_{0,2}(-e_{(-1,-1,0)} + e_{(0,-1,-1)} + e_{(0,1,1)} - e_{(1,1,0)}) , \\ u_{0,3} &= B_{0,2}(-e_{(-1,0,-1)} - e_{(0,-1,-1)} - e_{(0,1,1)} - e_{(1,0,1)}) ; \end{aligned} \quad (5.8)$$

$$\begin{aligned} u_{1,1} &= \frac{2i}{3} B_{0,4}(e_{(-2,-1,-1)} + e_{(-1,-2,-1)} - e_{(-1,-1,-2)} + e_{(1,1,2)} - e_{(1,2,1)} - e_{(2,1,1)}) \quad (5.9) \\ &\quad + \frac{2i}{3} B_{0,6}(-e_{(-2,-1,-1)} - e_{(-1,-2,-1)} + e_{(-1,-1,-2)} - e_{(1,1,2)} + e_{(1,2,1)} + e_{(2,1,1)}) , \\ u_{1,2} &= \frac{2i}{3} B_{0,4}(-e_{(-2,-1,-1)} - e_{(-1,-2,-1)} - e_{(-1,-1,-2)} + e_{(1,1,2)} + e_{(1,2,1)} + e_{(2,1,1)}) \\ &\quad + \frac{2i}{3} B_{0,6}(e_{(-2,-1,-1)} + e_{(-1,-2,-1)} + e_{(-1,-1,-2)} - e_{(1,1,2)} - e_{(1,2,1)} - e_{(2,1,1)}) , \\ u_{1,3} &= \frac{2i}{3} B_{0,4}(-e_{(-2,-1,-1)} + e_{(-1,-2,-1)} + e_{(-1,-1,-2)} - e_{(1,1,2)} - e_{(1,2,1)} + e_{(2,1,1)}) \\ &\quad + \frac{2i}{3} B_{0,6}(e_{(-2,-1,-1)} - e_{(-1,-2,-1)} - e_{(-1,-1,-2)} + e_{(1,1,2)} + e_{(1,2,1)} - e_{(2,1,1)}) . \end{aligned}$$

The next step is to compute the differential error $du^1/dt - \Delta u^1 - R\mathcal{P}(u^1, u^1)$ (which can be expressed via (3.7)) and the (time dependent) norms

$$\mathcal{D}_3 := \|u^1\|_3, \quad \mathcal{D}_4 := \|u^1\|_4, \quad \epsilon_3 := \left\| \frac{du^1}{dt} - \Delta u^1 - R\mathcal{P}(u^1, u^1) \right\|_3 . \quad (5.10)$$

These are as follows:

$$\mathcal{D}_3 = 4\sqrt{6} (2\pi)^{3/2} \left[B_{0,4} + 18R^2(B_{0,8} - 2B_{0,10} + B_{0,12}) \right]^{1/2} \quad (5.11)$$

$$\mathcal{D}_4 = 8\sqrt{3} (2\pi)^{3/2} \left[B_{0,4} + 54R^2(B_{0,8} - 2B_{0,10} + B_{0,12}) \right]^{1/2} , \quad (5.12)$$

$$\begin{aligned} \epsilon_3 = 8\sqrt{2/3} (2\pi)^{3/2} R^2 & \left[1065(B_{0,12} - 2B_{0,14} + B_{0,16}) \right. \\ & \left. + 3872R^2(B_{0,16} - 4B_{0,18} + 6B_{0,20} - 4B_{0,22} + B_{0,24}) \right]^{1/2}. \end{aligned} \quad (5.13)$$

The above functions determine the control Cauchy problem (5.5). The numerical solution of this problem for many sample values of R yields a picture as in items (i)(ii), page 17, with a critical Reynolds number

$$R_* \in (0.08, 0.09). \quad (5.14)$$

In Boxes 1a-1d we consider the case $R = 0.08$, giving information on the following functions of time: the quantity $(2\pi)^{3/2}|u_k^1(t)|$ for the wave vector $k = (1, 1, 0)$; the estimators \mathcal{D}_3 and ϵ_3 ; the solution \mathcal{R}_3 of the control Cauchy problem, which is global. In Boxes 2a-2d we consider the analogous functions in the case $R = 0.09$, where \mathcal{R}_3 diverges at $T_c = 2.153\dots$ ⁽⁷⁾. Each one of these boxes (and of the subsequent ones) contains the graph of the function under consideration, and its numerical values for some choices of t .

One immediately notices that the functions in boxes of the types (a) and (b) (i.e., the norm of the Fourier component $(1, 1, 0)$ and the estimator \mathcal{D}_3) are very similar in these examples (and in all the subsequent ones), even from the quantitative viewpoint. What really makes the difference among these examples (and the forthcoming ones) are the differential error estimator ϵ_3 and the solution \mathcal{R}_3 of the control Cauchy problem, considered in type (c) and (d) boxes.

Case $N = 2$. This relies on the function

$$u^1 := u_0 + Ru_1 + R^2u_2 : [0, +\infty) \rightarrow \mathbb{H}_{\Sigma^0}^5. \quad (5.15)$$

Of course, u_0, u_1 are as in Eqs. (5.8) (5.9); $u_2 = (u_{2,1}, u_{2,2}, u_{2,3})$ has a more complicated expression, and in order to save room we only report its first component. This is

$$\begin{aligned} u_{2,1} = & \frac{1}{9}B_{0,2}(-e_{(-1,-1,0)} - e_{(-1,0,-1)} - e_{(1,0,1)} - e_{(1,1,0)}) \\ & + \frac{1}{3}B_{0,4}(e_{(0,-2,0)} - e_{(0,0,-2)} - e_{(0,0,2)} + e_{(0,2,0)}) \\ & + \frac{1}{12}B_{0,6}(-e_{(-3,-2,-1)} - e_{(-3,-1,-2)} + e_{(-2,-3,-1)} - e_{(-2,-1,-3)} - e_{(-1,-3,-2)} - e_{(-1,-2,-3)} \\ & + 4e_{(-1,-1,0)} + 4e_{(-1,0,-1)} - 8e_{(0,-2,0)} + 8e_{(0,0,-2)} + 8e_{(0,0,2)} - 8e_{(0,2,0)} \\ & + 4e_{(1,0,1)} + 4e_{(1,1,0)} - e_{(1,2,3)} - e_{(1,3,2)} - e_{(2,1,3)} + e_{(2,3,1)} - e_{(3,1,2)} - e_{(3,2,1)}) \end{aligned} \quad (5.16)$$

⁷Here and in the sequel, an expression like $r = a.bcd\dots$ means that $a.bcd\dots$ are the first digits of the MATHEMATICA output in the computation of r .

$$\begin{aligned}
& + \frac{1}{9} B_{0,8} (e_{(-3,-2,-1)} + e_{(-3,-1,-2)} - e_{(-2,-3,-1)} + e_{(-2,-1,-3)} + e_{(-1,-3,-2)} + e_{(-1,-2,-3)} \\
& \quad - 2e_{(-1,-1,0)} - 2e_{(-1,0,-1)} + 3e_{(0,-2,0)} - 3e_{(0,0,-2)} - 3e_{(0,0,2)} + 3e_{(0,2,0)} \\
& \quad - 2e_{(1,0,1)} - 2e_{(1,1,0)} + e_{(1,2,3)} + e_{(1,3,2)} + e_{(2,1,3)} - e_{(2,3,1)} + e_{(3,1,2)} + e_{(3,2,1)}) \\
& + \frac{1}{36} B_{0,14} (-e_{(-3,-2,-1)} - e_{(-3,-1,-2)} + e_{(-2,-3,-1)} - e_{(-2,-1,-3)} - e_{(-1,-3,-2)} - e_{(-1,-2,-3)} \\
& \quad - e_{(1,2,3)} - e_{(1,3,2)} - e_{(2,1,3)} + e_{(2,3,1)} - e_{(3,1,2)} - e_{(3,2,1)}) .
\end{aligned}$$

The next step involves the differential error $du^2/dt - \Delta u^2 - R\mathcal{P}(u^2, u^2)$ and the (time dependent) norms

$$\mathcal{D}_3 := \|u^2\|_3, \quad \mathcal{D}_4 := \|u^2\|_4, \quad \epsilon_3 := \left\| \frac{du^2}{dt} - \Delta u^2 - R\mathcal{P}(u^2, u^2) \right\|_3 . \quad (5.17)$$

For example, one has

$$\begin{aligned}
\mathcal{D}_3 = & \frac{\sqrt{2}}{3\sqrt{3}} (2\pi)^{3/2} \left[1296 B_{0,4} + 288 R^2 (-B_{0,4} + 84 B_{0,8} - 164 B_{0,10} + 81 B_{0,12}) \right. \\
& \left. + R^4 (16 B_{0,4} + 1056 B_{0,8} - 4544 B_{0,10} + 16317 B_{0,12} - 29496 B_{0,14} + 17680 B_{0,16} \right. \\
& \left. + 6174 B_{0,20} - 8232 B_{0,22} + 1029 B_{0,28}) \right]^{1/2} . \quad (5.18)
\end{aligned}$$

The expressions of \mathcal{D}_4 and ϵ_3 are not reported. The former has a complexity similar to that of \mathcal{D}_3 , the latter is lengthier; in fact, ϵ_3 has the form $(2\pi)^{3/2} (\sum_{j,b} C_{j,b} R^j B_{0,b})^{1/2}$, where $C_{j,b}$ are rational coefficients and the sum involves 48 pairs (j, b) , with $j \in \{6, 8, 10\}$.

Let us pass to the control Cauchy problem (5.5); in the present case, the picture of items (i)(ii), page 17 is realized with a critical Reynolds number

$$R_* \in (0.12, 0.13) . \quad (5.19)$$

Boxes 3a-3d are about the case $R = 0.12$, and give information on the functions already chosen for the previous tables; one of them is the solution \mathcal{R}_3 of the control Cauchy problem, which is global. Boxes 4a-4d are about the analogous functions in the case $R = 0.13$, in which \mathcal{R}_3 diverges at $T_c = 2.604\dots$

Case $N = 5$. This relies on the function

$$u^5 := \sum_{j=0}^5 R^j u_j : [0, +\infty) \rightarrow \mathbb{H}_{\Sigma_0}^5 . \quad (5.20)$$

The terms u_0, u_1, u_2 are as before; the functions u_3, u_4 and u_5 have expressions of increasing complexity, that we cannot reproduce here. We only mention that each

one of the components $u_{5,1}, u_{5,2}, u_{5,3}$ is a linear combination with rational coefficients of 1924 terms of the form $iB_{a,b}e_k$, with $a \in \{0, 1, 2\}$; the wave vectors k appearing at least in one of the three components of u_5 are 174.

The differential error $du^5/dt - \Delta u^5 - R\mathcal{P}(u^5, u^5)$ and the norms

$$\mathcal{D}_3 := \|u^5\|_3, \quad \mathcal{D}_4 := \|u^5\|_4, \quad \epsilon_3 := \left\| \frac{du^5}{dt} - \Delta u^5 - R\mathcal{P}(u^5, u^5) \right\|_3 \quad (5.21)$$

have very lengthy expressions. Each one of the functions (5.21) has the form $(2\pi)^{3/2}(\sum_{j,a,b} C_{j,a,b} R^j B_{a,b})^{1/2}$, where the $C_{j,a,b}$ are rational coefficients and the sum involves finitely many triples (j, a, b) of nonnegative integers; these triples are 204 in the cases of \mathcal{D}_3 and \mathcal{D}_4 , and 1734 in the case of ϵ_3 .

As in the other cases, the final step is the control problem (5.5); the picture outlined by items (i)(ii), page 17 is now realized with a critical Reynolds number

$$R_* \in (0.23, 0.24) . \quad (5.22)$$

In Boxes 5a-5d and 6a-6d we give some information about the cases $R = 0.23$ and $R = 0.24$, respectively. Boxes 7a-7d are about the case $R = 0.12$; this choice is considered as well in the next subsection, where it is used for a comparison between the Reynolds expansion and the Galerkin approach.

Comparison with the Galerkin approach. In [14] we have considered the NS Cauchy problem with the BNW initial datum, using the Galerkin approximate solution u^G that corresponds to a finite set of Fourier modes $G \subset \mathbf{Z}^3 \setminus \{0\}$.

This approach relies on the (finite-dimensional) Galerkin subspace

$$\begin{aligned} \mathbb{H}_{\Sigma_0}^G &:= \{v \in \mathbb{D}'_{\Sigma_0} \mid v_k = 0 \text{ for } k \notin G\} \\ &= \left\{ \sum_{k \in G} v_k e_k \mid v_k \in \mathbf{C}^d, k \bullet v_k = 0 \text{ for all } k \in G \right\} \end{aligned} \quad (5.23)$$

and on the projection

$$\mathfrak{P}^G : \mathbb{D}'_{\Sigma_0} \rightarrow \mathbb{H}_{\Sigma_0}^G, \quad v = \sum_{k \in \mathbf{Z}^d \setminus \{0\}} v_k e_k \mapsto \mathfrak{P}^G v := \sum_{k \in G} v_k e_k . \quad (5.24)$$

The Galerkin approximate solution of the Cauchy problem (5.2) (with the BNW datum) corresponding to the set of modes G is the unique function u^G such that

$$u^G \in C^1([0, +\infty), \mathbb{H}_{\Sigma_0}^G), \quad \frac{du^G}{dt} = \Delta u^G + R \mathfrak{P}^G \mathcal{P}(u^G, u^G), \quad u^G(0) = \mathfrak{P}^G u_* \quad (5.25)$$

(thus, u^G solves a finite-dimensional Cauchy problem; one can show the existence of a global solution, of domain $[0, +\infty)$, using the fact that the L^2 norm is a decreasing

function of time). Indeed, in [14] we considered an equivalent formulation of (5.25) based on the viscosity ν and on the unscaled time \mathfrak{t} , related to R and to the present time variable t via Eq. (1.3); in the sequel we will rephrase the results of [14] in terms of the variables R and t .

For a given G , u^G can be computed numerically; more precisely, one solves numerically a system of ODEs for the Fourier coefficients $\hat{\gamma}_k$ in the expansion

$$u^G(t) = \sum_{k \in G} \hat{\gamma}_k(t) e_k . \quad (5.26)$$

One can specialize to u^G the general framework for approximate NS solutions; in particular, one introduces the tautological estimators of orders 3 or 4 for the growth and the differential error of u^G , i.e., the functions

$$\begin{aligned} \widehat{\mathcal{D}}_3(t) &:= \|u^G(t)\|_3 , \quad \widehat{\mathcal{D}}_4(t) := \|u^G(t)\|_4 , \\ \widehat{\epsilon}_3(t) &:= \left\| \left(\frac{du^G}{dt} - \Delta u^G - R\mathcal{P}(u^G, u^G) \right)(t) \right\|_3 \quad (t \in [0, +\infty)) \end{aligned} \quad (5.27)$$

(see [14] for an explicit expression of $\widehat{\epsilon}_3$ in terms of the Fourier coefficients $\hat{\gamma}_k$). The datum error $u^G(0) - u_*$ is zero if G contains the Fourier modes $\pm a, \pm b, \pm c$ involved in Eq. (5.4). Assuming this, one can analyze the Galerkin approximate solution in terms of a control Cauchy problem

$$\frac{d\widehat{\mathcal{R}}_3}{dt} = -\widehat{\mathcal{R}}_3 + R(G_3\widehat{\mathcal{D}}_3 + K_3\widehat{\mathcal{D}}_4)\widehat{\mathcal{R}}_3 + R G_3\widehat{\mathcal{R}}_3^2 + \widehat{\epsilon}_3 \text{ on } [0, \widehat{T}_c), \quad \widehat{\mathcal{R}}_3(0) = 0 , \quad (5.28)$$

whose (maximal) solution $\widehat{\mathcal{R}}_n \in C^1([0, \widehat{T}_c), [0, +\infty))$ can be computed numerically. In [14] we have employed a set G of 150 modes, of the following form:

$$G := S \cup -S ; \quad -S := \{-k \mid k \in S\} ; \quad (5.29)$$

$$\begin{aligned} S := & \{(0, 0, 2), (0, 1, -3), (0, 1, 1), (0, 1, 3), (0, 2, 0), (0, 2, 2), (0, 3, -1), (0, 3, 1), (0, 3, 3), \\ & (1, -3, -2), (1, -3, 0), (1, -3, 2), (1, -2, -3), (1, -2, -1), (1, -2, 1), (1, -2, 3), \\ & (1, -1, -2), (1, -1, 2), (1, 0, -3), (1, 0, 1), (1, 0, 3), (1, 1, -2), (1, 1, 0), (1, 1, 2), (1, 2, -3), \\ & (1, 2, -1), (1, 2, 1), (1, 2, 3), (1, 3, -2), (1, 3, 0), (1, 3, 2), (2, -3, -3), (2, -3, -1), (2, -3, 1), \\ & (2, -3, 3), (2, -2, -2), (2, -2, 2), (2, -1, -3), (2, -1, -1), (2, -1, 1), (2, -1, 3), (2, 0, 0), \\ & (2, 0, 2), (2, 1, -3), (2, 1, -1), (2, 1, 1), (2, 1, 3), (2, 2, -2), (2, 2, 0), (2, 3, -3), (2, 3, -1), \\ & (2, 3, 1), (2, 3, 3), (3, -3, -2), (3, -3, 2), (3, -2, -3), (3, -2, -1), (3, -2, 1), (3, -2, 3), \\ & (3, -1, -2), (3, -1, 0), (3, -1, 2), (3, 0, -1), (3, 0, 1), (3, 0, 3), (3, 1, -2), (3, 1, 0), (3, 1, 2), \\ & (3, 2, -3), (3, 2, -1), (3, 2, 1), (3, 2, 3), (3, 3, -2), (3, 3, 0), (3, 3, 2)\} . \end{aligned}$$

Let us summarize the results arising from this choice of G .

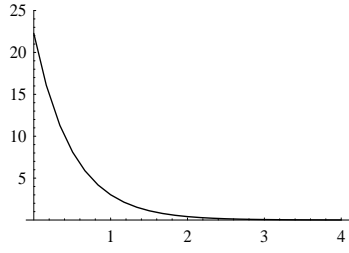
- (\hat{i}) There are indications for the existence of a critical Reynolds number R_* such that: $\hat{T}_c = +\infty$ and $\hat{\mathcal{R}}_3$ vanishes at $+\infty$ if $0 \leq R \leq R_*$, while $\hat{T}_c < +\infty$ and $\hat{\mathcal{R}}_3$ diverges at \hat{T}_c if $R > R_*$. One has $R_* \in (0.13, 0.14)$ ⁽⁸⁾.
- (\hat{ii}) Let us consider the maximal solution u of the NS Cauchy problem (5.2). For $0 \leq R \leq R_*$, (\hat{i}) grants that u is global and $\|u(t) - u^G(t)\|_3 \leq \hat{\mathcal{R}}_3(t)$ for all $t \in [0, +\infty)$. For $R > R_*$ (\hat{i}) only grants that the domain of u contains the interval $[0, \hat{T}_c)$, and that $\|u(t) - u^G(t)\|_3 \leq \hat{\mathcal{R}}_3(t)$ for all $t \in [0, \hat{T}_c)$.

As an example, Boxes 8a-8d report the main results about the Galerkin approach for $R = 0.12$. Both the general picture (\hat{i}) (\hat{ii}) and the results for $R = 0.12$ indicate a substantial equivalence between the Galerkin approach with the set G of (5.29) and the Reynolds expansion of order $N = 2$. In fact:

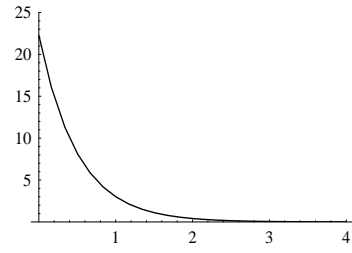
- (a) The expansion of order $N = 2$ yields a picture similar to (\hat{i}) (\hat{ii}), with a critical Reynolds number $R_* \in (0.12, 0.13)$ (see items (i)(ii) of page 17 and Eq. (5.19) for R_*).
- (b) The error estimators ϵ_3 , $\hat{\epsilon}_3$ and the solutions \mathcal{R}_3 , $\hat{\mathcal{R}}_3$ of the control equations for the $N = 2$ Reynolds expansion and for the Galerkin approach with G in (5.29) have similar orders of magnitude in the case $R = 0.12$, here used as a test to make comparisons: see Boxes 3c-3d and 8c-8d for more detailed information.

If we pass to the $N = 5$ expansion, we find a significant improvement with respect to the results of the above Galerkin approach. Let us recall that, for $N = 5$, the critical Reynolds number for global existence is in the interval $(0.23, 0.24)$; moreover, if we use again the case $R = 0.12$ for a comparison, we see that the error estimators ϵ_3 and the function \mathcal{R}_3 of the $N = 5$ expansion are much smaller than the homologous function $\hat{\epsilon}_3$, $\hat{\mathcal{R}}_3$ of the Galerkin approach: see Boxes 7c-7d and 8c-8d. In particular, the ratio $\mathcal{R}_3(t)/\hat{\mathcal{R}}_3(t)$ is of order 10^{-4} : so, the bound $\|u(t) - u^5(t)\|_3 \leq \mathcal{R}_3(t)$ is much more stringent than the bound $\|u(t) - u^G(t)\|_3 \leq \hat{\mathcal{R}}_3(t)$.

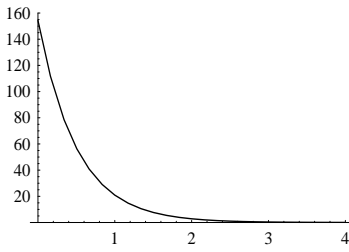
⁸The estimate coming directly from [14] is $1/8 < R_* < 1/7$, whence $0.125 < R_* < 0.143$; this estimate has been refined to $0.13 < R_* < 0.14$ by a supplementary run of the MATHEMATICA program for [14].



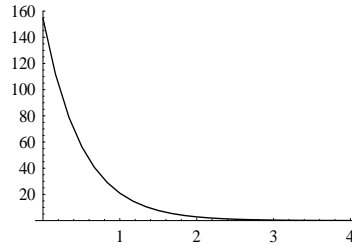
Box 1a. $N = 1, R = 0.08$: the function $\gamma(t) := (2\pi)^{3/2} |u_{(1,1,0)}^1(t)|$. One has $\gamma(0) = 22.27\dots$, $\gamma(0.5) = 8.189\dots$, $\gamma(1) = 3.012\dots$, $\gamma(1.5) = 1.108\dots$, $\gamma(2) = 0.4076\dots$, $\gamma(4) = 7.466\dots \times 10^{-3}$, $\gamma(8) = 2.504\dots \times 10^{-6}$, $\gamma(10) = 4.587\dots \times 10^{-8}$.



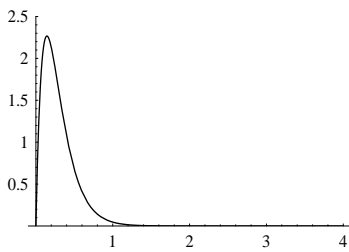
Box 2a. $N = 1, R = 0.09$: the function $\gamma(t) := (2\pi)^{3/2} |u_{(1,1,0)}^1(t)|$. One has $\gamma(0) = 22.27\dots$, $\gamma(0.5) = 8.188\dots$, $\gamma(1) = 3.011\dots$, $\gamma(1.5) = 1.107\dots$, $\gamma(2) = 0.4075\dots$, $\gamma(4) = 7.465\dots \times 10^{-3}$.



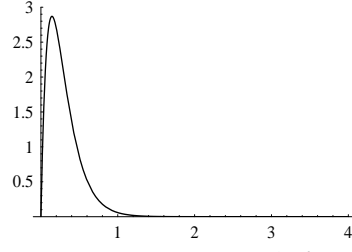
Box 1b. $N = 1, R = 0.08$: the function $D_3(t) := \|u^1(t)\|_3$. One has $D_3(0) = 154.3\dots$, $D_3(0.5) = 56.94\dots$, $D_3(1) = 20.90\dots$, $D_3(1.5) = 7.683\dots$, $D_3(2) = 2.826\dots$, $D_3(4) = 0.05176\dots$, $D_3(8) = 1.736\dots \times 10^{-5}$, $D_3(10) = 3.180\dots \times 10^{-7}$.



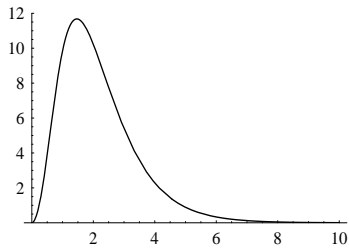
Box 2b. $N = 1, R = 0.09$: the function $D_3(t) := \|u^1(t)\|_3$. One has $D_3(0) = 154.3\dots$, $D_3(0.5) = 56.99\dots$, $D_3(1) = 20.90\dots$, $D_3(1.5) = 7.684\dots$, $D_3(2) = 2.826\dots$, $D_3(4) = 0.05176\dots$.



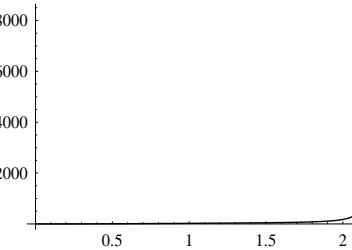
Box 1c. $N = 1, R = 0.08$: the function $\epsilon_3(t)$. One has $\epsilon_3(0) = 0$, $\epsilon_3(0.14) = 2.266\dots$, $\epsilon_3(1) = 0.04605\dots$, $\epsilon_3(2) = 1.296\dots \times 10^{-4}$, $\epsilon_3(4) = 8.108\dots \times 10^{-10}$.



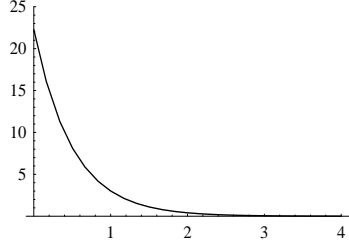
Box 2c. $N = 1, R = 0.09$: the function $\epsilon_3(t)$. One has $\epsilon_3(0) = 0$, $\epsilon_3(0.14) = 2.868\dots$, $\epsilon_3(1) = 0.05829\dots$, $\epsilon_3(2) = 1.640\dots \times 10^{-4}$, $\epsilon_3(4) = 1.026\dots \times 10^{-9}$.



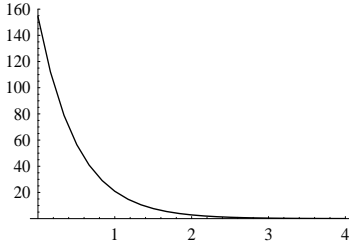
Box 1d. $N = 1, R = 0.08$: the function $\mathcal{R}_3(t)$. This appears to be globally defined, and vanishing at $+\infty$. One has $\mathcal{R}_3(0) = 0$, $\mathcal{R}_3(1) = 9.858\dots$, $\mathcal{R}_3(1.5) = 11.66\dots$, $\mathcal{R}_3(2) = 10.23\dots$, $\mathcal{R}_3(4) = 2.283\dots$, $\mathcal{R}_3(8) = 0.04547\dots$, $\mathcal{R}_3(10) = 6.162\dots \times 10^{-3}$.



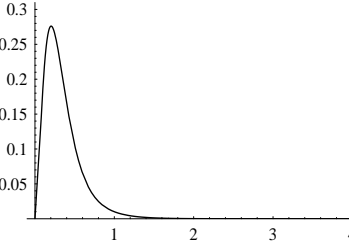
Box 2d. $N = 1, R = 0.09$: the function $\mathcal{R}_3(t)$. This diverges for $t \rightarrow T_c = 2.153\dots$. One has $\mathcal{R}_3(0) = 0$, $\mathcal{R}_3(0.5) = 6.624\dots$, $\mathcal{R}_3(1) = 23.20\dots$, $\mathcal{R}_3(1.5) = 46.60\dots$, $\mathcal{R}_3(2) = 176.0\dots$.



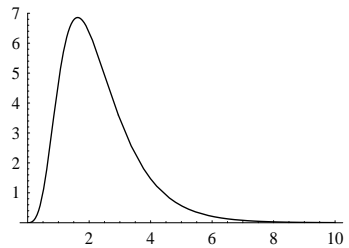
Box 3a. $N = 2, R = 0.12$: the function $\gamma(t) := (2\pi)^{3/2} |u_{(1,1,0)}^1(t)|$. One has $\gamma(0) = 22.27\dots$, $\gamma(0.5) = 8.184\dots$, $\gamma(1) = 3.009\dots$, $\gamma(1.5) = 1.107\dots$, $\gamma(2) = 0.4072\dots$, $\gamma(4) = 7.459\dots \times 10^{-3}$, $\gamma(8) = 2.502\dots \times 10^{-6}$, $\gamma(10) = 4.583\dots \times 10^{-8}$.



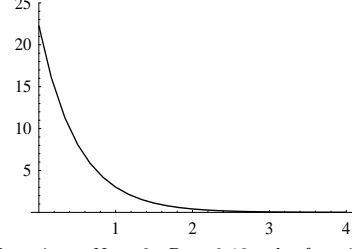
Box 3b. $N = 2, R = 0.12$: the function $\mathcal{D}_3(t) := \|u^1(t)\|_3$. One has $\mathcal{D}_3(0) = 154.3\dots$, $\mathcal{D}_3(0.5) = 57.10\dots$, $\mathcal{D}_3(1) = 20.88\dots$, $\mathcal{D}_3(1.5) = 7.672\dots$, $\mathcal{D}_3(2) = 2.821\dots$, $\mathcal{D}_3(4) = 0.05168\dots$, $\mathcal{D}_3(8) = 1.733\dots \times 10^{-5}$, $\mathcal{D}_3(10) = 3.175\dots \times 10^{-7}$.



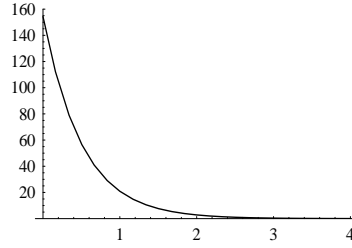
Box 3c. $N = 2, R = 0.12$: the function $\epsilon_3(t)$. One has $\epsilon_3(0) = 0$, $\epsilon_3(0.20) = 0.2759\dots$, $\epsilon_3(1) = 0.01007\dots$, $\epsilon_3(2) = 1.688\dots \times 10^{-4}$, $\epsilon_3(4) = 5.653\dots \times 10^{-8}$.



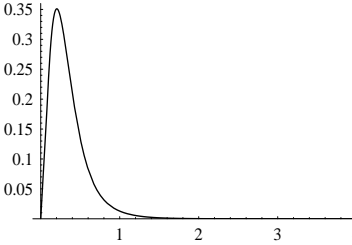
Box 3d. $N = 2, R = 0.12$: the function $\mathcal{R}_3(t)$. This appears to be globally defined, and vanishing at $+\infty$. One has $\mathcal{R}_3(0) = 0$, $\mathcal{R}_3(1) = 4.699\dots$, $\mathcal{R}_3(1.6) = 6.854\dots$, $\mathcal{R}_3(2) = 6.368\dots$, $\mathcal{R}_3(4) = 1.466\dots$, $\mathcal{R}_3(8) = 0.02914\dots$, $\mathcal{R}_3(10) = 3.949\dots \times 10^{-3}$.



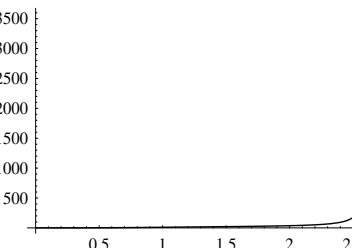
Box 4a. $N = 2, R = 0.13$: the function $\gamma(t) := (2\pi)^{3/2} |u_{(1,1,0)}^1(t)|$. One has $\gamma(0) = 22.27\dots$, $\gamma(0.5) = 8.183\dots$, $\gamma(1) = 3.009\dots$, $\gamma(1.5) = 1.106\dots$, $\gamma(2) = 0.4071\dots$, $\gamma(4) = 7.457\dots \times 10^{-3}$.



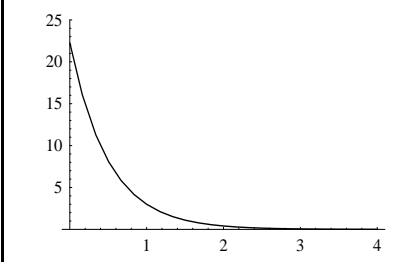
Box 4b. $N = 2, R = 0.13$: the function $\mathcal{D}_3(t) := \|u^1(t)\|_3$. One has $\mathcal{D}_3(0) = 154.3\dots$, $\mathcal{D}_3(0.5) = 57.16\dots$, $\mathcal{D}_3(1) = 20.89\dots$, $\mathcal{D}_3(1.5) = 7.671\dots$, $\mathcal{D}_3(2) = 2.821\dots$, $\mathcal{D}_3(4) = 0.05166\dots$.



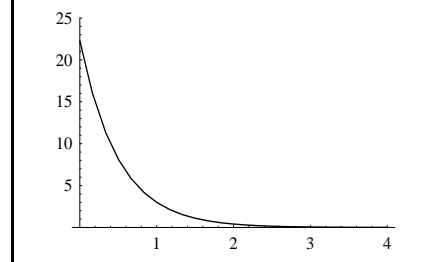
Box 4c. $N = 2, R = 0.13$: the function $\epsilon_3(t)$. One has $\epsilon_3(0) = 0$, $\epsilon_3(0.20) = 0.3508\dots$, $\epsilon_3(1) = 0.01280\dots$, $\epsilon_3(2) = 2.146\dots \times 10^{-4}$, $\epsilon_3(4) = 7.187\dots \times 10^{-8}$.



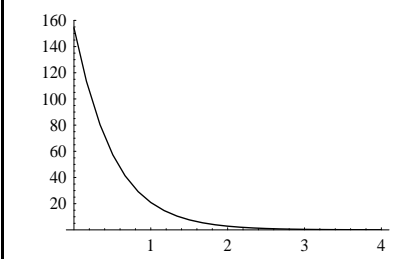
Box 4d. $N = 2, R = 0.13$: the function $\mathcal{R}_3(t)$. This diverges for $t \rightarrow T_c = 2.604\dots$. One has $\mathcal{R}_3(0) = 0$, $\mathcal{R}_3(0.5) = 1.735\dots$, $\mathcal{R}_3(1) = 10.20\dots$, $\mathcal{R}_3(1.5) = 20.68\dots$, $\mathcal{R}_3(2) = 36.30\dots$, $\mathcal{R}_3(2.5) = 175.3$.



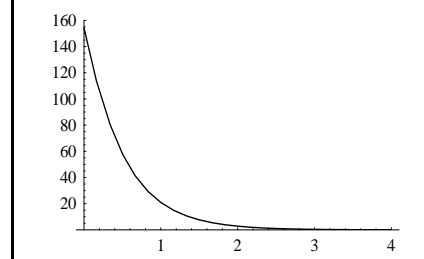
Box 5a. $N = 5, R = 0.23$: the function $\gamma(t) := (2\pi)^{3/2} |u_{(1,1,0)}^5(t)|$. One has $\gamma(0) = 22.27\dots$, $\gamma(0.5) = 8.160\dots$, $\gamma(1) = 2.997\dots$, $\gamma(1.5) = 1.102\dots$, $\gamma(2) = 0.4055\dots$, $\gamma(4) = 7.428\dots \times 10^{-3}$, $\gamma(8) = 2.491\dots \times 10^{-6}$, $\gamma(10) = 4.564\dots \times 10^{-8}$.



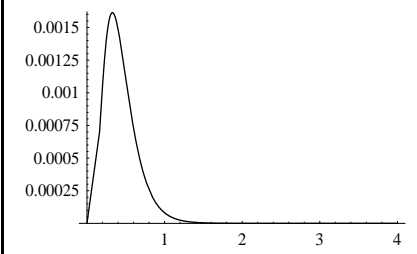
Box 6a. $N = 5, R = 0.24$: the function $\gamma(t) := (2\pi)^{3/2} |u_{(1,1,0)}^5(t)|$. One has $\gamma(0) = 22.27\dots$, $\gamma(0.5) = 8.157\dots$, $\gamma(1) = 2.996\dots$, $\gamma(1.5) = 1.101\dots$, $\gamma(2) = 0.4053\dots$, $\gamma(3) = 0.05485\dots$, $\gamma(4) = 7.424\dots \times 10^{-3}$.



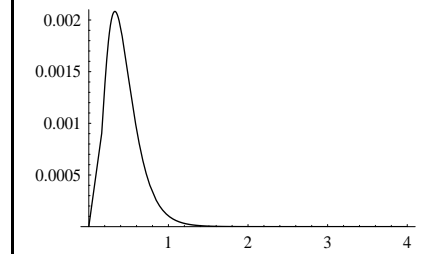
Box 5b. $N = 5, R = 0.23$: the function $\mathcal{D}_3(t) = \|u^5(t)\|_3$. One has $\mathcal{D}_3(0) = 154.3\dots$, $\mathcal{D}_3(0.5) = 56.97\dots$, $\mathcal{D}_3(1) = 20.90\dots$, $\mathcal{D}_3(1.5) = 7.646\dots$, $\mathcal{D}_3(2) = 2.810\dots$, $\mathcal{D}_3(4) = 0.05146\dots$, $\mathcal{D}_3(8) = 1.726\dots \times 10^{-5}$, $\mathcal{D}_3(10) = 3.162\dots \times 10^{-7}$.



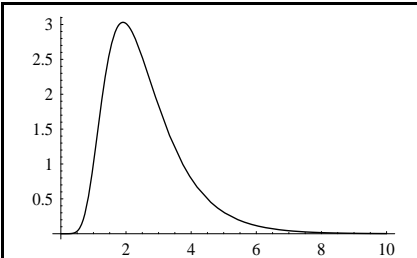
Box 6b. $N = 5, R = 0.24$: the function $\mathcal{D}_3(t) = \|u^5(t)\|_3$. One has $\mathcal{D}_3(0) = 154.3\dots$, $\mathcal{D}_3(0.5) = 58.08\dots$, $\mathcal{D}_3(1) = 20.90\dots$, $\mathcal{D}_3(1.5) = 7.643\dots$, $\mathcal{D}_3(2) = 2.808\dots$, $\mathcal{D}_3(3) = 0.3800$, $\mathcal{D}_3(4) = 0.05143\dots$.



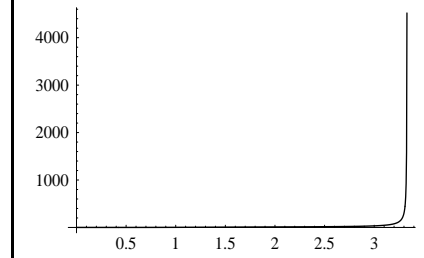
Box 5c. $N = 5, R = 0.23$: the function $\epsilon_3(t)$. One has $\epsilon_3(0) = 0$, $\epsilon_3(0.33) = 1.613\dots \times 10^{-3}$, $\epsilon_3(2) = 2.061\dots \times 10^{-7}$, $\epsilon_3(3) = 8.234 \times 10^{-10}$, $\epsilon_3(4) = 1.182\dots \times 10^{-11}$.



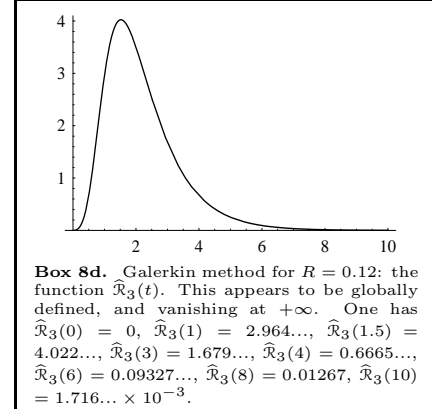
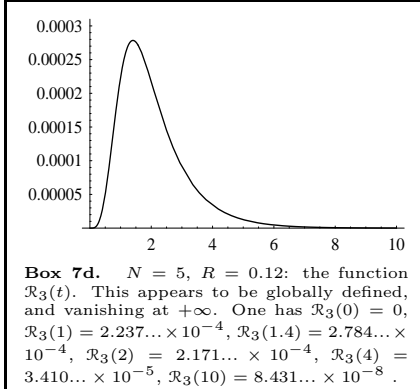
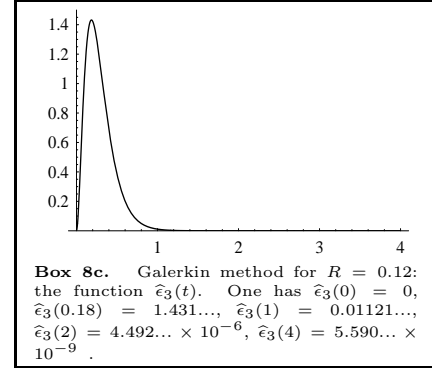
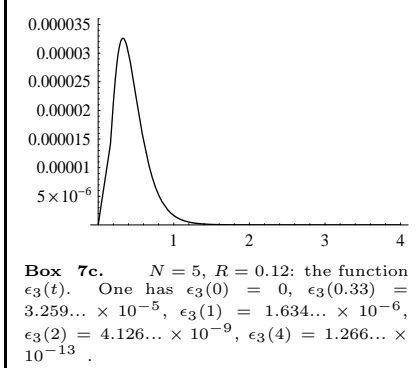
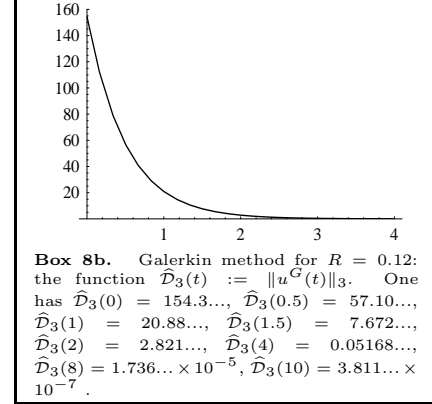
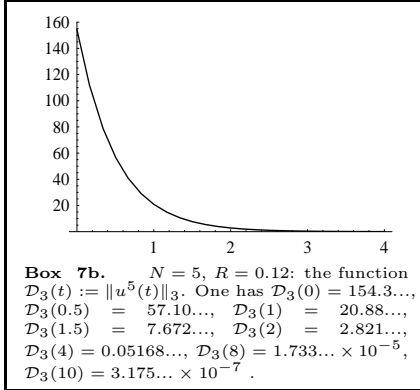
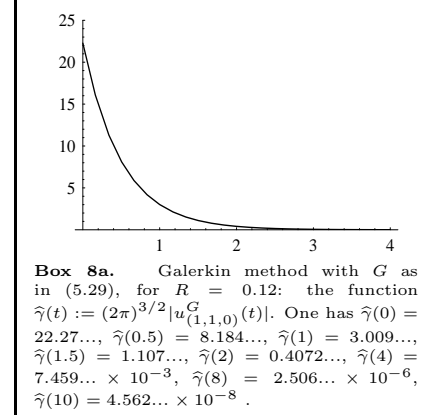
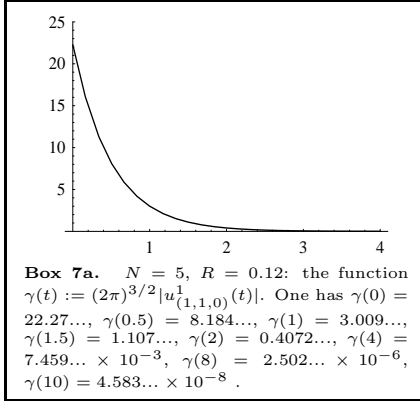
Box 6c. $N = 5, R = 0.24$: the function $\epsilon_3(t)$. One has $\epsilon_3(0) = 0$, $\epsilon_3(0.33) = 2.0827 \times 10^{-3}$, $\epsilon_3(1) = 1.043 \times 10^{-4}$, $\epsilon_3(2) = 2.664\dots \times 10^{-7}$, $\epsilon_3(4) = 1.591\dots \times 10^{-11}$.



Box 5d. $N = 5, R = 0.23$: the function $\mathcal{R}_3(t)$. This appears to be globally defined, and vanishing at $+\infty$. One has $\mathcal{R}_3(0) = 0$, $\mathcal{R}_3(1) = 1.004\dots$, $\mathcal{R}_3(1.5) = 2.609\dots$, $\mathcal{R}_3(2) = 3.014\dots$, $\mathcal{R}_3(4) = 0.7907\dots$, $\mathcal{R}_3(8) = 0.01580\dots$, $\mathcal{R}_3(10) = 2.141\dots \times 10^{-3}$.



Box 6d. $N = 5, R = 0.24$: the function $\mathcal{R}_3(t)$. This diverges for $t \rightarrow T_c = 3.332\dots$. One has $\mathcal{R}_3(0) = 0$, $\mathcal{R}_3(0.5) = 0.06348\dots$, $\mathcal{R}_3(1) = 2.126\dots$, $\mathcal{R}_3(2) = 10.89\dots$, $\mathcal{R}_3(3) = 33.31\dots$, $\mathcal{R}_3(3.33) = 4520.7\dots$.



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References

- [1] E. Behr, J. Nečas, H. Wu, *On blow-up of solution for Euler equations*, M2AN: Math. Model. Numer. Anal. **35** (2001), 229-238.
- [2] M. Cannone, *Ondelettes, paraproducts et Navier-Stokes*, Diderot, 1995.
- [3] S.I. Chernyshenko, P. Constantin, J.C. Robinson, E.S. Titi, *A posteriori regularity of the three-dimensional Navier-Stokes equations from numerical computations*, J. Math. Phys. **48** (2007), 065204/10.
- [4] T.Kato, *Nonstationary flows of viscous and ideal fluids in \mathbf{R}^3* , J.Funct.Anal. **9** (1972), 296-305.
- [5] T. Kato, *Quasi-linear equations of evolution, with applications to partial differential equations*, in “Spectral theory and differential equations”, Proceedings of the Dundee Symposium, Lecture Notes in Mathematics **448** (1975), 23-70.
- [6] S. Kida, *Three-dimensional periodic flows with high-symmetry*, J. Phys. Soc. Japan **54** (1985), 2132-2140.
- [7] P.G. Lemarié-Rieusset, “Recent developments in the Navier-Stokes problem”, Chapman & Hall, Boca Raton (2002).
- [8] C. Morosi, M. Pernici, L. Pizzocchero, *On power series solutions for the Euler equation, and the Behr-Nečas-Wu initial datum*, ESAIM Math. Model. Numer. Anal. **47** (2013), 663-688. See also arXiv:1203.6865
- [9] C. Morosi, M. Pernici, L. Pizzocchero, *A posteriori estimates for Euler and Navier-Stokes equations*, submitted (2013).
- [10] C. Morosi, L. Pizzocchero, *On approximate solutions of semilinear evolution equations*, Rev. Math. Phys. **16** (2004), 383-420. See also arXiv:math-ph/0309016
- [11] C. Morosi, L. Pizzocchero, *On approximate solutions of semilinear evolution equations II. Generalizations, and applications to Navier-Stokes equations*, Rev. Math. Phys. **20** (2008), 625-706. See also arXiv:0709.1670

- [12] C. Morosi, L. Pizzocchero, *An H^1 setting for the Navier-Stokes equations: Quantitative estimates*, Nonlinear Anal. **74** (2011), 2398-2414. See also arXiv:0909.3707
- [13] C. Morosi, L. Pizzocchero, *On the constants in a Kato inequality for the Euler and NS equations*, Commun. Pure Appl. Analysis **11**(2012), 557-586. See also arXiv:1009.2051
- [14] C. Morosi, L. Pizzocchero, *On approximate solutions for the Euler and Navier-Stokes equations*, Nonlinear Analysis **75** (2012), 2209-2235. See also arXiv:1104.3832
- [15] C. Morosi, L. Pizzocchero, *On the constants in a basic inequality for the Euler and NS equations*, Appl. Math. Lett. **26** (2013), 277-284. See also arXiv:1007.4412
- [16] C. Morosi, M. Pernici, L. Pizzocchero, in preparation.
- [17] F. W. J. Olver *et al.*, editors, *NIST Handbook of Mathematical Functions*, Cambridge University Press, 2010.
- [18] J.C. Robinson, W. Sadowski, *Numerical verification of regularity in the three-dimensional Navier-Stokes equations for bounded sets of initial data*, Asymptot. Anal. **59** (2008), 39-50.
- [19] Y. Sinai, *Power series for solutions of the 3D Navier-Stokes system on \mathbf{R}^3* , J. Stat. Phys. **121** (2005), 779–803.
- [20] G.I. Taylor, A.E. Green, *Mechanism of the production of small eddies from large ones*, Proc. R. Soc. Lond. A **158** (1937), 499-521.